# Math 372

November 28, 2022

# Today

Goal: Distributive lattices.

# Leftover from previous class

**Theorem.** (Least action principle) Let $G$ be finite, and let $D \in \mathrm{Div}(G)$. Suppose that $v_1, \ldots, v_k$ is a sequence of legal vertex firings for $D$, and let $\sigma = \sum_{i=1}^{k} v_i$. Then if $D - L\tau$ is stable with $\tau \geq 0$, it follows that $\tau \geq \sigma$.

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**Corollary.** Let $G$ be finite and $D \in \mathrm{Div}(G)$. Suppose that $v_1, \ldots, v_k$ and $w_1, \ldots, w_\ell$ are both legal firing sequences, and let $\sigma := \sum_{i=}^{k} v_i$ and $\tau := \sum_{i=1}^{\ell} w_i$ be the corresponding firing scripts. Further suppose that $D \xrightarrow{\sigma} D'$ and $D \xrightarrow{\tau} D''$ with both $D'$ and $D''$ stable. Then $\sigma = \tau$ and $D' = D''$.

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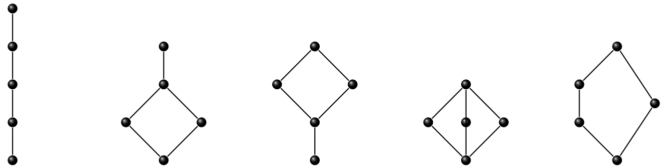
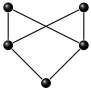
Hasse diagrams for all lattices with five elements:



# Distributive lattices

Not a lattice:



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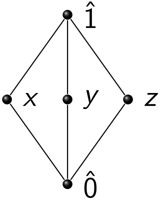
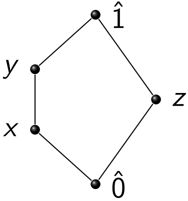
Two non-distributive lattices:




Diamond lattice, $M_3$

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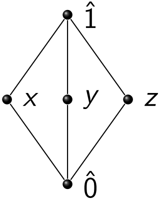


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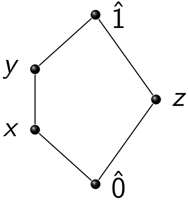


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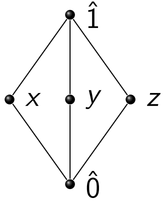


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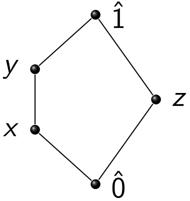


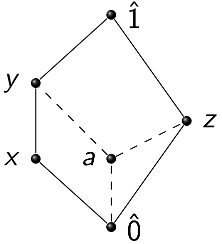

Pentagon lattice, $N_5$

**Proposition.** A lattice is distributive if and only if none of its sublattices is isomorphic to $M_3$ or $N_5$.

*Sublattice*: subset that is closed under the meet and join operations of the original lattice.

# Distributive lattices

Example of a distributive lattice that contains $N_5$ as a subset but not as a sublattice:




A distributive lattice.

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join = union, and meet = intersection

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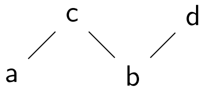


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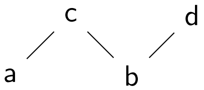


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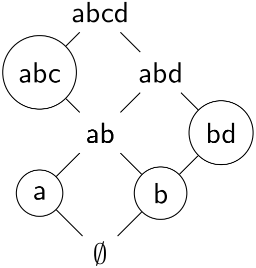


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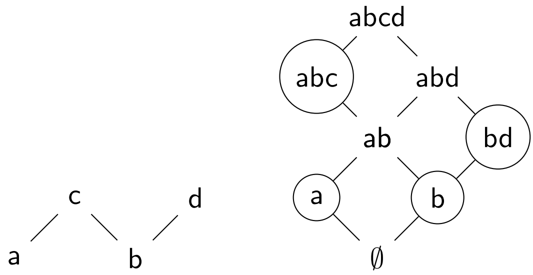


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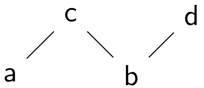


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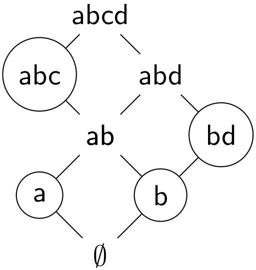


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$x$ in a lattice is *join irreducible* if $x \neq \widehat{0}$ and if it is not possible to write $x = y \vee z$ with $y < x$ and $z < x$ ($x$ covers exactly one element).

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Then the firing graph $\mathcal{F}(D)$ has no cycles, and when considered as a poset, $\mathcal{F}(D) \simeq J_f(P)$.

**Proof.** Exercise.

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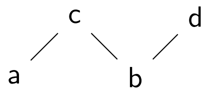


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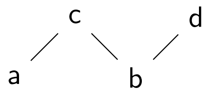


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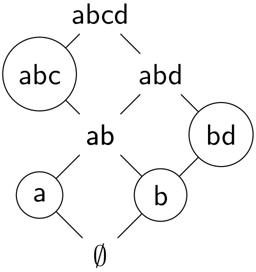


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