Math 372

November 16, 2022

Today

Goal: Firing graphs.

Set-up

Assumptions on $G = (V, E)$:

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- *finitary:* finite number of edges incident on each vertex
- there exists a finite path between any two vertices.

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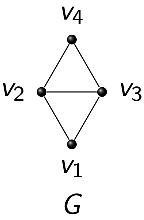
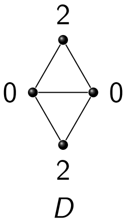
Firing vertex v:

$$D' = D - Lv.$$

Coordinates





$$D = 2v_1 + 2v_4 \quad \leftrightsquigarrow \quad (1, 0, 0, 1)$$

Firing graph

Definition. The *firing graph* for $D \in \mathrm{Div}(G)$ is the directed graph $\mathcal{F}(D)$ whose vertices are the divisors reachable from D by a finite sequence of legal vertex-firings, and with an edge from vertex H to vertex H' if there is a legal vertex firing taking H to H'.

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However, since $D(w) \leq m$ for all $w \in V$, the above equality can only hold if $D(w) = m$ for all w adjacent to v. Since G is connected, it must be that $D(w) = m$ for all $w \in V$. $\square$

Cycles in the firing graph

Corollary. Suppose G is finite, and let $D \in \mathrm{Div}(G)$.

1. Let C be a directed cycle in $\mathcal{F}(D)$ starting and ending at some divisor H. Say

$$H = H_1 \xrightarrow{v_1} H_2 \xrightarrow{v_2} \cdots \xrightarrow{v_k} H_{k+1} = H$$

is a sequence of legal vertex firings corresponding to C. Then $\sum_{i=1}^{k} v_i = a\mathbf{1}_V$ for some integer a.

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2. Suppose $v_1, v_1, \ldots, v_k$ and $w_1, \ldots, w_\ell$ are both legal firing sequences taking D to the some divisor D'. Also suppose that neither firing sequence contains all of the vertices. Then the sequences are the same up to a permutation.

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3. The firing poset is graded. The rank of $D' \in \mathcal{F}(D)$ is the number of vertices that must be fired from D to reach D' in the firing graph (which also equals the length of a smallest legal sequence of firings taking D to D').

Least action principle

Theorem. (Least action principle) Let G be finite, and let $D \in \mathrm{Div}(G)$. Suppose that $v_1, \ldots, v_k$ is a sequence of legal vertex firings for D, and let $\sigma = \sum_{i=1}^{k} v_i$. Then if $D - L\tau$ is stable with $\tau \geq 0$, it follows that $\tau \geq \sigma$.

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Then τ' stabilizes D' and $v_2, \ldots, v_k$ is a sequence of legal vertex firings. By induction $\sigma - v_1 \leq \tau'$, and the result follows:

$$\sigma \leq \tau' + v_1 = \tau.$$

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Corollary. Let G be finite and $D \in \mathrm{Div}(G)$. Suppose that $v_1, \ldots, v_k$ and $w_1, \ldots, w_\ell$ are both legal firing sequences, and let $\sigma := \sum_{i=}^{k} v_i$ and $\tau := \sum_{i=1}^{\ell} w_i$ be the corresponding firing scripts. Further suppose that $D \xrightarrow{\sigma} D'$ and $D \xrightarrow{\tau} D''$ with both D' and D'' stable. Then $\sigma = \tau$ and $D' = D''$.