Math 372

November 16, 2022

Today

Goal: Matrix-tree theorem.

Directed graphs

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The columns of L encode the firing rules in a chip-firing game on G.

Directed spanning trees

Definition. A *(directed) spanning tree of G rooted at $s \in V$* is a subgraph T such that for all $v \in V$, there exists a unique directed path in T from v to s. The vertex s is the *root* or *sink* of the tree.

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- T contains *all* of the vertices of G (hence, the word "spanning");
- T contains no directed cycles;
- for all vertices v of G, the outdegree of v in T is 0 if $v = s$, and is 1, otherwise. In particular, T contains no multiple edges.

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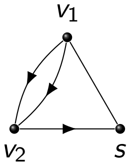

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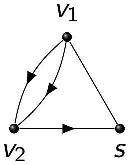

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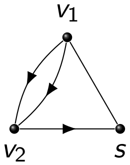

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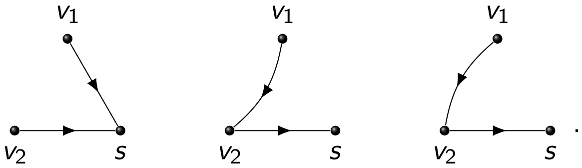
Spanning trees rooted at s:



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$$\widetilde{L} = \begin{pmatrix} \sum_{i\neq 1} a_{1i} & -a_{21} & -a_{31} & \dots & -a_{n-1,1} \\ -a_{12} & \sum_{i\neq 2} a_{2i} & -a_{32} & \dots & -a_{n-1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1,n-1} & -a_{2,n-1} & -a_{3,n-1} & \dots & \sum_{i\neq n-1} a_{n-1,i} \end{pmatrix}$$

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$\sum_{i \neq \ell}$ means the sum over $i \in \{1, \dots, n\} \setminus \{\ell\}$

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$$\widetilde{L}_{\sigma(k),k} = \begin{cases} \sum_{i\neq k} a_{k,i} & \text{if } k\in\mathrm{Fix}(\sigma) \\ -a_{k,\sigma(k)} & \text{otherwise.} \end{cases}$$

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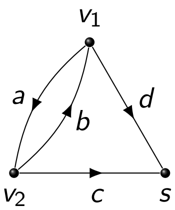
Example



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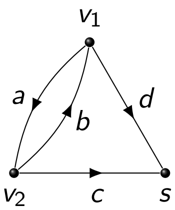

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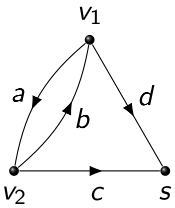

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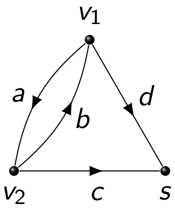

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&\quad = (-1)(a_{1,2} + \cdots + a_{1,10})(-a_{2,7})(-a_{3,5})(a_{4,1} + \cdots + a_{4,10})' \\
&\qquad \cdot (-a_{5,9})(a_{6,1} + \cdots + a_{6,10})'(-a_{7,2})(a_{8,1} + \cdots + a_{8,10})'(-a_{9,3}),
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\end{aligned}$$

$$\begin{aligned}
&= (-1)\Big[\overbrace{(a_{1,2} + \dots)}^{\sigma(1)=1}\overbrace{(a_{4,1} + \dots)'}^{\sigma(4)=4}\overbrace{(a_{6,1} + \dots)'}^{\sigma(6)=6}\overbrace{(a_{8,1} + \dots)'}^{\sigma(8)=8}\Big] \\
&\quad \cdot \Big[\underbrace{(-a_{2,7})(-a_{7,2})}_{(2,7)}\underbrace{(-a_{3,5})(-a_{5,9})(-a_{9,3})}_{(3,5,9)}\Big].
\end{aligned}$$

$$(-1)[\overbrace{(a_{1,2}+\dots)}^{\sigma(1)=1}\overbrace{(a_{4,1}+\dots)'}^{\sigma(4)=4}\overbrace{(a_{6,1}+\dots)'}^{\sigma(6)=6}\overbrace{(a_{8,1}+\dots)'}^{\sigma(8)=8}]$$

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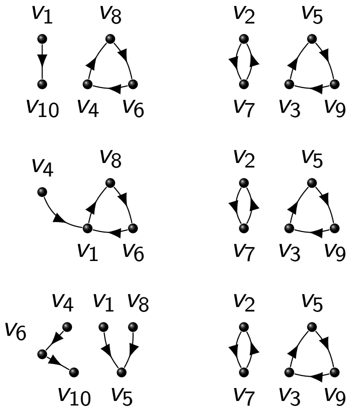
	F	C	monomial
(a)			$a_{1,10}\, a_{2,7}\, a_{3,5} \cdots a_{9,3}$
(b)			$a_{1,8}\, a_{2,7}\, a_{3,5} \cdots a_{9,3}$
(c)			$a_{1,5}\, a_{2,7}\, a_{3,5} \cdots a_{9,3}$

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Claim: The sign of a monomial corresponding to σ in the expansion is $(-1)^{\#\text{ non-trivial cycles of }\sigma}$.

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$$\operatorname{sgn}\big((2,7)(3,5,9)\big)\ \underbrace{a_{1,10}a_{4,8}a_{6,4}a_{8,6}}_{\mathrm{Fix}(\sigma)=\{1,4,6,8\}}\ \underbrace{(-a_{2,7})(-a_{7,2})}_{(2,7)}\underbrace{(-a_{3,5})(-a_{5,9})(-a_{9,3})}_{(3,5,9)}$$

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Claim: The sign of a monomial corresponding to σ in the expansion is $(-1)^{\#\text{ non-trivial cycles of }\sigma}$.

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$$\operatorname{sgn}((2,7)(3,5,9))\ \underbrace{a_{1,10}a_{4,8}a_{6,4}a_{8,6}}_{\operatorname{Fix}(\sigma)=\{1,4,6,8\}}\ \underbrace{(-a_{2,7})(-a_{7,2})}_{(2,7)}\underbrace{(-a_{3,5})(-a_{5,9})(-a_{9,3})}_{(3,5,9)}$$

$$\operatorname{sgn}((2,7))(-a_{2,7})(-a_{7,2}) = -1\cdot a_{2,7}a_{7,2}$$
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