Math 372

November 14, 2022

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via $(a,b,c,d,e) \mapsto (a,b,d,e)$.

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When M is regarded as a linear function, these row and column operations correspond to integer changes of bases on the codomain and domain of M, respectively.

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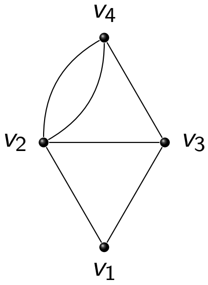
- $\det U = \pm 1$ and $\det V = \pm 1$,
- and $UMV = D$.

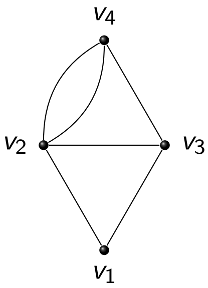
Example

Determine the structure of $\mathrm{Pic}(G)$ for the graph pictured below:



Example

Determine the structure of $\mathrm{Pic}(G)$ for the graph pictured below:



Recall $\mathrm{Pic}(G) \approx \mathrm{cok}(L)$ where L is the Laplacian matrix,

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix}.$$

Example

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix} \xrightarrow{c_1 \to c_1 + c_2} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 3 & 4 & -1 & -2 \\ -2 & -1 & 3 & -1 \\ -2 & -2 & -1 & 3 \end{pmatrix}$$

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$$\xrightarrow[c_3 \to c_3 + c_1]{c_2 \to c_2 + c_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 7 & 2 & -2 \\ -2 & -3 & 1 & -1 \\ -2 & -4 & -3 & 3 \end{pmatrix}$$

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$$ULV = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Therefore,

$$\mathrm{Pic}(G) = \mathsf{cok}(L) \simeq \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z}.$$

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$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{U} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ -3a + b \\ -16a + 6b + c \\ a + b + c + d \end{pmatrix}$$

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and so $\mathrm{Jac}(G) \simeq \mathbb{Z}/13\mathbb{Z}$.

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- $a + (b + c) = (a + b) + c$ for all $a, b, c \in A$.

Z-modules

If A is an abelian group, $a \in A$, and $n \in \mathbb{N}$, define

$$na = \underbrace{a + \cdots + a}_{n \text{ times}}$$

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Conversely, every $\mathbb{Z}$-module is an abelian group.

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Define a $\mathbb{Z}$-linear mapping

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This mapping is surjective (why?) Let K be the kernel of this mapping. Then we have a short exact sequence

$$0 \to K \to \mathbb{Z}^m \to A \to 0.$$

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By a course in algebra, K is finitely generated, say with n generators. So as with A, there is a surjective mapping $\mathbb{Z}^n \to K$:

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\mathbb{Z}^n \arrow[rr, dashed, "M"] \arrow[dr, two heads] & & \mathbb{Z}^m \arrow[r] & A \arrow[r] & 0 \\
& K \arrow[ur, hook] & & &
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The induced mapping $\mathbb{Z}^n \to \mathbb{Z}^m$ is represented by a matrix M, and

$$A \approx \mathrm{cok}(M).$$

Structure theorem for finitely-generated abelian groups

Suppose U and V are matrices, invertible over $\mathbb{Z}$ such that $UMV = N$. Then we have a commutative diagram inducing an isomorphism $\operatorname{cok}(M) \to \operatorname{cok}(N)$:

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\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "M"] \arrow[d, "V^{-1}"', "\wr"] & \mathbb{Z}^m \arrow[r] \arrow[d, "U", "\wr"'] & \operatorname{cok} M \arrow[r] \arrow[d, dashed] & 0 \\
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\mathbb{Z}^n \arrow[r, "N"] & \mathbb{Z}^m \arrow[r] & \mathrm{cok}\, N \arrow[r] & 0.
\end{tikzcd} \tag{1}$$

We can always choose U and V so that $N = UMV$ is diagonal.

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Theorem. (Structure theorem for f.g. abelian groups) A group is a finitely generated abelian group if and only if it is isomorphic to

$$\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z} \times \mathbb{Z}^r$$

for some list (possibly empty) of integers $n_1, \ldots, n_k$ with $n_i > 1$ for all i and some integer $r \geq 0$.

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Condition 1: $n_i | n_{i+1}$ (n_i evenly divides n_{i+1}) for all i. In this case, the n_i are the *invariant factors* of the group.

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Condition 1: $n_i | n_{i+1}$ (n_i evenly divides n_{i+1}) for all i. In this case, the n_i are the *invariant factors* of the group.

Condition 2: There exist primes $p_1 \leq \cdots \leq p_k$ and positive integers m_i such that $n_i = p_i^{m_i}$ for all i. In this case, the n_i are the *elementary divisors* and the $\mathbb{Z}/n_i\mathbb{Z}$ are the *primary factors* of the group.

Structure theorem for finitely-generated abelian groups

Theorem. (Structure theorem for f.g. abelian groups) A group is a finitely generated abelian group if and only if it is isomorphic to

$$\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z} \times \mathbb{Z}^r$$

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The number r is the *rank* of the group.

Structure theorem for finitely-generated abelian groups

Theorem. (Chinese remainder theorem.) Let $m, n \in \mathbb{Z}$. Then

$$\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

if and only if m and n are relatively prime. If $\gcd(m, n) = 1$, then an isomorphism is provided by $a \mapsto (a \bmod m, a \bmod n)$.

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$$\mathbb{Z}/24\mathbb{Z} \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

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Examples.

$$\mathbb{Z}/24\mathbb{Z} \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/4\mathbb{Z} \not\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Structure theorem for finitely-generated abelian groups

Definition. An $m \times n$ integer matrix M is in *Smith normal form* if

$$M = \mathrm{diag}(s_1, \ldots, s_k, 0, \ldots, 0),$$

a diagonal matrix, where $s_1, \ldots, s_k$ are positive integers such that $s_i | s_{i+1}$ for all i. The s_i are called the *invariant factors* of M.

Structure theorem for finitely-generated abelian groups

Example. The matrix

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in Smith normal form with invariant factors $s_1 = 1$, $s_2 = 2$, and $s_3 = 12$.

Structure theorem for finitely-generated abelian groups

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is in Smith normal form with invariant factors $s_1 = 1$, $s_2 = 2$, and $s_3 = 12$. We have

$$\operatorname{cok}(M) := \mathbb{Z}^5 / \operatorname{im}(M) \simeq \mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}^2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}^2.$$

So $\operatorname{cok}(M)$ has rank $r = 2$ and its invariant factors are 2 and 12.