Math 372

November 11, 2022

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via $(a,b,c,d,e) \mapsto (a,b,d,e)$.

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When M is regarded as a linear function, these row and column operations correspond to integer changes of bases on the codomain and domain of M, respectively.

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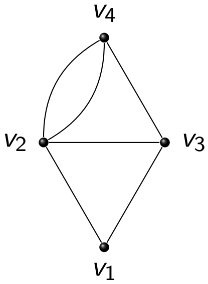
- $\det U = \pm 1$ and $\det V = \pm 1$,
- and $UMV = D$.

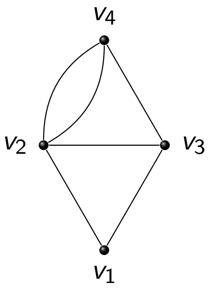
Example

Determine the structure of $\mathrm{Pic}(G)$ for the graph pictured below:



Example

Determine the structure of $\mathrm{Pic}(G)$ for the graph pictured below:



Recall $\mathrm{Pic}(G) \approx \mathsf{cok}(L)$ where L is the Laplacian matrix,

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix}.$$

Example

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix} \xrightarrow{c_1 \to c_1 + c_2} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 3 & 4 & -1 & -2 \\ -2 & -1 & 3 & -1 \\ -2 & -2 & -1 & 3 \end{pmatrix}$$

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$$\xrightarrow[c_3 \to c_3 + c_1]{c_2 \to c_2 + c_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 7 & 2 & -2 \\ -2 & -3 & 1 & -1 \\ -2 & -4 & -3 & 3 \end{pmatrix}$$

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$$ULV = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -1 & -2 \\ -1 & -1 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & -3 & 7 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Therefore,

$$\mathrm{Pic}(G) = \mathrm{cok}(L) \simeq \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z}/13\mathbb{Z}.$$

Example

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -16 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{U} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ -3a + b \\ -16a + 6b + c \\ a + b + c + d \end{pmatrix}$$

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and so $\mathrm{Jac}(G) \simeq \mathbb{Z}/13\mathbb{Z}$.