Math 361 problems for Friday, Week 9

Factoring rational integers in a number ring

For today's class, please work on the following problems. Some subset of them will be part of next week's HW.

PROBLEM 1. Let $f(x) = x^3 + bx^2 + cx + d$ be an irreducible cubic. Then the *discriminant* of $f$ is the discriminant of $\{1, \theta, \theta^2\}$ where $\theta$ is a root of $f$. It turns out to be given by the formula

$$\Delta[1, \theta, \theta^2] = b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd.$$

Consider the case $f(x) = x^3 + x^2 + 3$.

1. Prove that $f$ is irreducible.
2. Let $K = \mathbb{Q}(\theta)$ where $\theta$ is a root of $f$. Compute the discriminant of $f$, and use it to argue that $\mathfrak{O}_K = \mathbb{Z}[\theta]$. (We proved a theorem a while ago that will be useful.)
3. Factor the ideals $(p)$ of $\mathfrak{O}_K$ into primes for $p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$.
4. There is a theorem of Dedekind that says a rational prime $p$ *ramifies* if and only if $p$ divides the discriminant. Are your results from the previous part of this problem consistent with that result?

PROBLEM 2. Let $K = \mathbb{Q}(\sqrt{7})$. For primes $p$ such that $2 \leq p \leq 23$, use the theorem introduced in the last class (involving factoring the minimal polynomial of $\sqrt{7}$ modulo $p$) to factor the principal ideal $(p)$ of $\mathfrak{O}_K$. (Take a look at the Wikipedia page on quadratic reciprocity for some relevant material.)