Math 361 Homework for Friday, Week 11

Problem 1. In our proof of the two squares theorem, we constructed a lattice and applied Minkowski's theorem using a circle X of a certain radius to obtain a lattice point solving our problem. Consider that proof in the case $p = 5$. For that case, draw the lattice L, draw the circle X, and indicate the lattice point (a, b) such that $a^2 + b^2 = 5$.

Problem 2.

(a) Consider the lattice $L = \text{Span}\{(1, 1), (0, 1)\}$. Give two different fundamental domains for L and compute their areas.

(b) Let $L \subset \mathbb{R}^n$ be an arbitrary lattice, and let F_1 and F_2 be fundamental domains for L. Why is it the case that $\text{vol}(F_1) = \text{vol}(F_2)$? (You may assume that L has rank n for ease of exposition.)

Problem 3. Let $L \subset \mathbb{Z}^n$ be a rank n lattice, and let F be a fundamental domain for L. Why is it the case that $\text{vol}(F)$ is the number of elements in $\mathbb{Z}^n \cap F$? (Recall that just after discussing the Smith normal form, we discovered a formula for the number of elements in $\mathbb{Z}^n/L$.)

Problem 4. Verify the two squares theorem for all eleven primes less than 100 and congruent to 1 modulo 4.

Problem 5. Verify the four squares theorem for all integers n such that $75 \leq n \leq 79$.

Problem 6. Prove that not every positive integer is the sum of three squares.