Math 316 Homework for Friday, Week 9

Problem 1. For each of the following matrices M:

- Find matrices P, Q, and D such that P and Q are invertible over the integers, D is diagonal, and $PMQ = D$. (You do not need to show your work for this part.)
- Use the mapping P to create an explicit isomorphism of $\operatorname{cok}(M)$ with a product of cyclic groups.

(a) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 15 & 15 & 12 \\ 3 & -3 & 6 \\ 12 & 18 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 15 \\ 21 & 27 \\ 24 & 33 \end{pmatrix}$.

Problem 2. In $\mathbb{Z}[\sqrt{-6}]$ we have the factorizations

$$6 = 2 \cdot 3 = (\sqrt{-6})(-\sqrt{-6}).$$

Our goal is to present an idea that allows us to factor (6) into prime ideals. First notice that $-6 = 2(-3)$, and consider the ring $\mathbb{Z}[\sqrt{2}, \sqrt{-3}]$. In that ring, we have the factorization

$$6 = (-1)(\sqrt{2})^2(\sqrt{-3})^2.$$

Let $\mathfrak{a} = (\sqrt{2}) \subset \mathbb{Z}[\sqrt{2}, \sqrt{-3}]$ be the principal ideal in $\mathbb{Z}[\sqrt{2}, \sqrt{-3}]$ generated by $\sqrt{2}$, and let $\mathfrak{p} = (2, \sqrt{-6})$.

(a) Prove that $1 \notin \mathfrak{p}$.

(b) Show that the ideal $\mathfrak{p} = (2, \sqrt{-6})$ is a maximal ideal in $\mathbb{Z}[\sqrt{-6}]$, hence prime. [Hint: give an isomorphism of $\mathbb{Z}[\sqrt{-6}]/\mathfrak{p}$ and a field. Make sure you show your mapping is well-defined, surjective, and injective. Part ((a)) can help.]

(c) Prove that $\mathfrak{a} \cap \mathbb{Z}[\sqrt{-6}] \neq \mathbb{Z}[\sqrt{-6}]$ by showing that $1 \notin \mathfrak{a}$.

(d) Prove that $\mathfrak{p} = \mathfrak{a} \cap \mathbb{Z}[\sqrt{-6}]$. (The previous two parts of this problem may help.)

(e) Mimic the above construction with $\sqrt{-3}$ in place of $\sqrt{2}$ to define another maximal ideal $\mathfrak{q}$. (You do not need to turn in proofs, just produce the ideal $\mathfrak{q}$. You should check on scratch paper, though, that the same reasoning applies.)

(f) Prove that $(6) = \mathfrak{p}^2\mathfrak{q}^2$.

Problem 3. In $\mathbb{Z}[\sqrt{-10}]$, we have the factorizations

$$14 = 2 \cdot 7 = (2 + \sqrt{-10})(2 - \sqrt{-10}).$$

Mimic the technique of the previous problem to factor the principal ideal $(14) \subset \mathbb{Z}[\sqrt{-10}]$ into prime ideals. Use the technique to guess the ideals, then prove the ideals are maximal, and check the factorization. Outline the ideas involved.