Math 316 Homework for Friday, Week 7

Problem 1. Let P be an ideal in the ring R. Prove the following:

(a) P is prime if and only if for all ideals I and J such $IJ \subseteq P$, we have $I \subseteq P$ or $J \subseteq P$.

(b) P is maximal if and only if R/P is a field. (Don't forget the detail that a maximal ideal is a proper subset of R.)

Problem 2. Let $\mathbb{Z}_{(2)} = \{a/b : a, b \in \mathbb{Z} \text{ with } (a,b) = 1 \text{ and } b \text{ odd}\}$. Define addition and multiplication as usual addition and multiplication in $\mathbb{Q}$ followed by cancellation of common factors in the numerator and denominator. Note that $0 = 0/1$ and $1 = 1/1$ are elements of $\mathbb{Z}_{(2)}$.

(a) Show that $\mathbb{Z}_{(2)}$ is a domain by showing it is closed under addition and multiplication, and it has no zero divisors.

(b) Describe all of the units of $\mathbb{Z}_{(2)}$.

(c) Describe all of the ideals of $\mathbb{Z}_{(2)}$. (You should find that $\mathbb{Z}_{(2)}$ is a PID. The following notation may help: for a nonzero $a \in \mathbb{Z}_{(2)}$, let $\nu_2(a) = k$ if $a = 2^k a'$ with a' is a unit of $\mathbb{Z}_{(2)}$.)

(d) Describe all of the prime ideals and all of the maximal ideals of $\mathbb{Z}_{(2)}$.

(e) Describe all of the irreducibles of $\mathbb{Z}_{(2)}$ up to multiplication by units.

(f) Is $\mathbb{Z}_{(2)}$ a UFD? If so, describe the factorization of a nonzero, non-unit into irreducibles.

(g) Consider the following "proof" that the ring of integers $\mathfrak{O}_K$ in any number field K has infinitely many pairwise non-associate irreducibles? (Recall, two irreducibles are associates if they differ by a unit factor.) The proof is by contradiction. Suppose there are finitely many pairwise non-associated irreducible $p_1, \ldots, p_k$. Define

$$r = 1 + p_1 \cdots p_k.$$

Since $\mathfrak{O}_K$ is a factorization domain, r can be factored into irreducibles. None of these irreducibles are among the $p_1, \ldots, p_k$. So there must be another irreducible.

Show that this problem is incorrect by applying it to $\mathbb{Z}_{(2)}$ instead of $\mathfrak{O}_K$. (Nevertheless, it is true that every number ring contains infinitely many irreducibles. We will see how to prove this later.)