Math 316 Homework for Friday, Week 6

Problem 1.

(a) Is $\mathbb{Z}[\frac{1}{2}]$ a finitely generated $\mathbb{Z}$-module? Prove or disprove.

(b) What is wrong with the following argument: $\mathbb{Z}$ is a PID, hence, Noetherian. By the Hilbert basis theorem, it follows that $\mathbb{Z}[x]$ is Noetherian. We have a surjective homomorphism $\mathbb{Z}[x] \to \mathbb{Z}[\frac{1}{2}]$ determined by $x \mapsto \frac{1}{2}$. Since $\mathbb{Z}[\frac{1}{2}]$ is the image of something that is Noetherian, it is Noetherian, hence, finitely generated.

Problem 2. Prove that $\mathbb{Z}[x]$ is not a PID by giving a specific example of an ideal that is not generated by a single element. Prove your example is not principal. (Note: this problem allows us to conclude that $\mathbb{Z}[x]$ is not a Euclidean domain with respect to any function.)

Problem 3. Let R be a Euclidean domain with respect to the function $d\colon R \setminus \{0\} \to \mathbb{N}$.

(a) Show that $d(1) \leq d(r)$ for all $r \in R \setminus \{0\}$.

(b) Show that $u \in R \setminus \{0\}$ is a unit if and only if $d(u) = d(1)$.

Problem 4. Consider the ideal $I = (3, 2+\sqrt{-5})$ in the ring of integers $\mathbb{Z}[\sqrt{-5}] \subset \mathbb{Q}(\sqrt{-5})$. We will show $\mathbb{Z}[\sqrt{-5}]$ in not a Euclidean domain by showing I is not principal.

(a) Show $I \neq \mathbb{Z}[\sqrt{-5}]$ by showing $1 \notin I$.

(b) Find all of the elements of $\mathbb{Z}[\sqrt{-5}]$ having norms ± 1, ± 3, and ± 9.

(c) Suppose that $I = (\alpha)$ for some $\alpha \in \mathbb{Z}[\sqrt{-5}]$, and find a contradiction.

Problem 5. In the problem, we prove that $\mathbb{Z}[\sqrt{3}]$ is a Euclidean domain with respect to the function $d(\alpha) := |N(\alpha)|$. Let $\alpha, \beta \in \mathbb{Z}[\sqrt{3}] \setminus \{0\}$.

(a) We have $d\colon \mathbb{Z}[\sqrt{3}] \setminus \{0\} \to \mathbb{N}$. Show that if $\alpha | \beta$, then $d(\alpha) \leq d(\beta)$.

(b) Show that in any case, there exist $q, r \in \mathbb{Z}[\sqrt{3}]$ such that

$$\beta = q\alpha + r$$

where $r = 0$ or $d(r) < d(a)$. To see this, first note that since $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}[\sqrt{3}]$ (or via the usual conjugation trick), $\frac{\beta}{\alpha} = s + t\sqrt{3}$ for some $s, t \in \mathbb{Q}$. Let $m, n \in \mathbb{Z}$ be integers closest to s and t, respectively. Thus, $|s - m| \leq 1/2$ and $|t - n| \leq 1/2$. Let $q = m + n\sqrt{3}$. Prove the result from this point. (Hint: you may need to use the fact that if $x, y \in \mathbb{R}_{\geq 0}$, then $|x - y| \leq \max\{x, y\}$.)