Math 316 Homework for Friday, Week 5

Problem 1.

(a) Let $K = \mathbb{Q}(\sqrt{-d})$ where $d \geq 1$ is a square-free integer and $-d \neq 1 \bmod 4$. Use norms to compute the units in $\mathfrak{O}_K$. (The case $d = 1$ will turn out to be an exception to the rule.)

(b) Now consider the number field $K = \mathbb{Q}(\sqrt{2})$. Find an infinite number of units in $\mathfrak{O}_K$. (Note that when α is a unit, so it α^k for all k.

Problem 2. Let M be an R-module. A *submodule* of M is a subset $N \subseteq M$ that is an R-module with the operations inherited from M. To prove $N \subseteq M$ is a submodule, show the following: (i) N is nonempty (usually done by showing that $0 \in N$, (ii) if $x, y \in N$, then $x + y \in N$, and (iii) if $r \in R$ and $x \in N$, then $rx \in N$.

(a) Let $N_1 \subseteq N_2 \subseteq \dots$ be an ascending chain of submodules of an R-module M. Show that $\cup_i N_i$ is an R-submodule of M.

(b) Let N and N' be submodules of an R-module M. Show that $N \cap N'$ is a submodule.

Problem 3. Find an ascending chain of maximal length of ideals in $\mathbb{Z}$ starting with the principal ideal (120).

Problem 4. Let K be a number field of degree n, and let $(\alpha_1, \dots, \alpha_n)$ be a $\mathbb{Q}$-basis for K. Let β be a nonzero element of K. Prove that

$$\Delta[\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n] = N(\beta)^2 \Delta[\alpha_1, \dots, \alpha_n].$$

Problem 5. Let $\alpha \in \mathbb{C}$ be a root of $p(x) = x^3 - x - 1$, and consider the number field $K = \mathbb{Q}(\alpha)$. We would like to show that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathfrak{O}_K$.

(a) Prove that p is irreducible.

(b) We have the formula

$$\Delta[1, \alpha, \alpha^2] = (-1)^{\binom{3}{2}} N(p'(\alpha)) = -N(p'(\alpha))$$

where p' is the derivative of p. Let $\beta := p'(\alpha)$. Our problem it to compute $N(\beta)$.

(i) Show that $\alpha = \frac{3}{\beta - 2}$. (Hint: we have $0 = p(\alpha) = \alpha^3 - \alpha - 1$. Multiply through by α^{-1}.) (Note that it follows that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, so $\mathbb{Q}(\beta)$ has degree 3, too. You will need this in the next part of this problem).

(ii) Write

$$p\left(\frac{3}{x-2}\right) = \frac{r(x)}{s(x)}$$

where $\deg r = \deg s = 3$ and r is monic. Argue that one of $r(x)$ is the minimal polynomial for β.

(iii) Use r to find the norm of β.

(iv) What is $\Delta[1, \alpha, \alpha^2]$. (Be careful with signs.)

(v) Why can we conclude that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathfrak{O}_K$?