Math 316 Homework for Friday, Week 4

Problem 1. For each of the following fields K, find an integral basis for $\mathfrak{O}_K$ and compute the discriminant of K.

(a) $\mathbb{Q}(\sqrt{44})$.

(b) $\mathbb{Q}(\sqrt{-44})$.

Problem 2. Carefully re-read the lecture notes for Friday, Week 3.

Let $\{\alpha_1, \ldots, \alpha_n\}$ is a $\mathbb{Q}$-basis for a number field K consisting of algebraic integers. Suppose that $\Delta[\alpha_1, \ldots, \alpha_n] = d$ where d is the discriminant of K. Using results from the Friday, Week 3 lecture, prove that $\{\alpha_1, \ldots, \alpha_n\}$ is a $\mathbb{Z}$-basis for $\mathfrak{O}_K$.

Problem 3. In this problem, we will prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. To start, note that in $\mathbb{Z}[\sqrt{-5}]$, we have the following two factorizations of 6.

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

Show that $2, 3, 1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are not units and are irreducible in $\mathbb{Z}[\sqrt{-5}]$. [Hints: Note that $\mathbb{Z}[\sqrt{-5}]$ is the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{-5})$. Let N be the norm on $\mathbb{Q}(\sqrt{-5})$. On Monday, Week 4, we saw that the (i) $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbb{Q}(\sqrt{-5})$, (ii) if $\alpha \in \mathfrak{O}_{\mathbb{Q}(\sqrt{-5})}$, then $N(\alpha) \in \mathbb{Z}$, and (iii) $\alpha \in \mathfrak{O}_{\mathbb{Q}(\sqrt{-5})}$ is a unit if and only if $N(\alpha) = \pm 1$. Use the norm to prove that the above four elements are irreducible. You will probably need to prove that 2 and 3 are never the norms of elements of $\sqrt{-5}$.)