Math 316 Homework for Friday, Week 3

Problem 1. Prove that $1/\sqrt{2}$ is not an algebraic integer.

Problem 2. Theorems, corollaries, and lemmas in this problem refer to the Wednesday Week 2 lecture. The numbers $\sqrt{3}$ and i are algebraic integers since they are roots of the monic polynomials $x^2 - 3$ and $x^2 + 1$, respectively. The purpose of this problem is to show that $\sqrt{3} + i$ is an algebraic integer by using the proof of Theorem 4 with $A = \mathbb{Z}$ and $B = A[\sqrt{3}, i]$.

(a) Find a $\mathbb{Z}$-module basis for $\mathbb{Z}[\sqrt{3}]$ and a $\mathbb{Z}[\sqrt{3}]$-module basis for $\mathbb{Z}[\sqrt{3}, i]$. No proof is required.

(b) Use your bases, and the ideas from Lemma 5 and Corollary 6 to find a $\mathbb{Z}$-module basis for $\mathbb{Z}[\sqrt{3}, i]$. Give a proof that there is no non-trivial $\mathbb{Z}$-linear relation among your basis elements.

(c) Use the $\mathbb{Z}$-module basis for $\mathbb{Z}[\sqrt{3}, i]$ from part (b) and the $(3 \Rightarrow 1)$ part of the proof of Theorem 4 to compute a monic polynomial p with integer coefficients such that $p(\sqrt{3} + i) = 0$. To find p, you will be taking the determinant of a 4×4 matrix.

Problem 3. Let K be a number field, i.e., a finite extension of the rationals. Show that if $\theta \in K$ then there exists a nonzero $c \in \mathbb{Z}$ and $\alpha \in \mathfrak{O}_K$ such that $c\theta = \alpha$. (Thoughts you should be having: What is a number field? Why does that imply θ is algebraic over $\mathbb{Q}$? Can the minimal polynomial for θ over $\mathbb{Q}$ help?) [Note that from this exercise, it follows that K is the quotient field of its integers, i.e., every element of K has the form α/β with α, β integers in K. From the primitive element theorem, it also follows that $K = \mathbb{Q}[\alpha]$ for some $\alpha \in \mathfrak{O}_K$.]

Problem 4. Let K be a number field, and let $I \subset \mathfrak{O}_K$ be a nonzero ideal.

(a) Show that there exists a nonzero $c \in I \cap \mathbb{Z}$. (Hint: Take $\eta \in I \setminus \{0\}$. Since η is an (algebraic) integer, its minimal polynomial has coefficients in $\mathbb{Z}$. Argue that this polynomial has a nonzero constant term and that this constant term is in I.)

(b) Show that there exists $\eta \in I$ such that $K = \mathbb{Q}[\eta]$. (Hint: by the primitive element theorem and the second problem in this homework set, we can write $K = \mathbb{Q}[\theta]$ for some $\theta \in \mathfrak{O}_K$. Now use part (a).)

Problem 5. Let $\alpha = 1 + 5^{1/3} + 5^{2/3}$. Show that α is an algebraic integer by using the determinant trick from Theorem 4, Wednesday Week 2, to find a monic polynomial with integer coefficients that vanishes at α. (Recall that proof that α is an integer iff there is a finitely generated $\mathbb{Z}$-module, M, such that $\alpha\, M \subseteq M$.)

Problem 6. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a basis for $K/\mathbb{Q}$. The four monomorphisms of K into $\mathbb{C}$ are determined by sending $\sqrt{2}$ to $\pm\sqrt{2}$ and, independently, sending $\sqrt{3} \to \pm\sqrt{3}$. Calculate the discriminant of this basis.