Math 361

April 10, 2023

Quiz

For Wednesday's quiz:

- What is a rank m *lattice* in $\mathbb{R}^n$?
- What is a *fundamental domain* for a lattice in $\mathbb{R}^n$?
- If L is a rank n lattice in $\mathbb{R}^n$, and $T^n = S^1 \times \cdots \times S^1$ is the n-torus, how is our standard mapping $\pi\colon \mathbb{R}^n/L \to T^n$ defined?
- With the above notation, if $Y \subseteq T^n$, how is the *volume* of Y defined?
- State and prove Minkowski's lattice point theorem for centrally symmetric convex sets centered at the origin.

Reminder

Please turn in idea(s) for your final project on Wednesday.

Today

- Lattice associated with a number field.
- Volume of fundamental region determined by $\mathfrak{a} \subseteq \mathfrak{O}_K$.

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where $z_j = u_j + v_j i \in \mathbb{C}$ for $j = 1, \ldots, t$. Then define

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Identifying $\mathbb{L}^{s,t}$ with $\mathbb{R}^n$, regard $\sigma(\mathfrak{a}) \subset \mathbb{R}^n$.

Then $\sigma(\mathfrak{a})$ is a lattice with fundamental domain of volume

$$2^{-t} N(\mathfrak{a}) \sqrt{|\Delta|}$$

where Δ is the discriminant of K.

Proof

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$$\sigma_k(\alpha_\ell) = \begin{cases} x_{k,\ell} & \text{if } 1 \leq k \leq s \\ u_{k,\ell} + iv_{k,\ell} & \text{if } s+1 \leq k \leq s+t \end{cases}$$

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We will show these $\mathbf{w}_\ell$ are $\mathbb{R}$-linearly independent. Hence, their $\mathbb{Z}$-span is a lattice. We will show the volume of a fundamental region is $2^{-t}\sqrt{|\Delta[\alpha_1, \dots, \alpha_n]|}$.

Proof

$$\det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{s,1} & \cdots & x_{s,n} \\ u_{s+1,1} & \cdots & u_{s+1,n} \\ v_{s+1,1} & \cdots & v_{s+1,n} \\ \vdots & \ddots & \vdots \\ u_{s+t,1} & \cdots & u_{s+t,n} \\ v_{s+t,1} & \cdots & v_{s+t,n} \end{pmatrix}$$

Proof

$$= \det \begin{pmatrix} \sigma_1(\alpha_1) & \cdots & \sigma_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_s(\alpha_1) & \cdots & \sigma_s(\alpha_n) \\ \frac{\sigma_{s+1}(\alpha_1)+\overline{\sigma}_{s+1}(\alpha_1)}{2} & \cdots & \frac{\sigma_{s+1}(\alpha_n)+\overline{\sigma}_{s+1}(\alpha_n)}{2} \\ \frac{\sigma_{s+1}(\alpha_1)-\overline{\sigma}_{s+1}(\alpha_1)}{2i} & \cdots & \frac{\sigma_{s+1}(\alpha_n)-\overline{\sigma}_{s+1}(\alpha_n)}{2i} \\ \vdots & \ddots & \vdots \\ \frac{\sigma_{s+t}(\alpha_1)+\overline{\sigma}_{s+t}(\alpha_1)}{2} & \cdots & \frac{\sigma_{s+t}(\alpha_n)+\overline{\sigma}_{s+1}(\alpha_n)}{2} \\ \frac{\sigma_{s+t}(\alpha_1)-\overline{\sigma}_{s+t}(\alpha_1)}{2i} & \cdots & \frac{\sigma_{s+t}(\alpha_n)-\overline{\sigma}_{s+t}(\alpha_n)}{2i} \end{pmatrix}$$

Proof

$$= \left(\frac{1}{2}\right)^{2t}\left(\frac{1}{i}\right)^{t}\det\begin{pmatrix} \sigma_1(\alpha_1) & \dots & \sigma_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_s(\alpha_1) & \dots & \sigma_s(\alpha_n) \\ \sigma_{s+1}(\alpha_1)+\overline{\sigma}_{s+1}(\alpha_1) & \dots & \sigma_{s+1}(\alpha_n)+\overline{\sigma}_{s+1}(\alpha_n) \\ \sigma_{s+1}(\alpha_1)-\overline{\sigma}_{s+1}(\alpha_1) & \dots & \sigma_{s+1}(\alpha_n)-\overline{\sigma}_{s+1}(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_{s+t}(\alpha_1)+\overline{\sigma}_{s+t}(\alpha_1) & \dots & \sigma_{s+t}(\alpha_n)+\overline{\sigma}_{s+1}(\alpha_n) \\ \sigma_{s+t}(\alpha_1)-\overline{\sigma}_{s+t}(\alpha_1) & \dots & \sigma_{s+t}(\alpha_n)-\overline{\sigma}_{s+t}(\alpha_n) \end{pmatrix}$$

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Example

Let $K = \mathbb{Q}(\sqrt{7})$, and let $\mathfrak{a} = (3, 1 + \sqrt{7})$ (this is one of the prime factors of $(3) \subset \mathfrak{O}_K$).

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$$(a + b\sqrt{7}) \cdot 3 + (c + d\sqrt{7})\sqrt{7} = (a + 3c + 7d) + (3b + c + d)\sqrt{7},$$

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So this set is the $\mathbb{Z}$-image of the matrix $\begin{pmatrix} 3 & 0 & 1 & 7 \\ 0 & 3 & 1 & 1 \end{pmatrix}$.

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