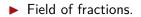
# Math 361

### March 8, 2023







Dedekind domains.



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The quotient field Q(R) of R is the field of fractions  $\{a/b : a \in R, b \in R \setminus \{0\}\}$ . It is the smallest field containing R.

$$R \hookrightarrow Q(R)$$
  
 $r \mapsto r/1.$ 

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Conversely, since K is a field and contains  $\mathfrak{O}_K$ , it contains the field of fractions of  $\mathfrak{O}_K$ .

**Definition.** A domain R is *integrally closed* if the only elements of its field of fractions Q(R) that are integral over R are the elements of R, itself.

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**Definition** A *Dedekind domain* is an integrally closed Noetherian domain in which every nonzero prime ideal is maximal.

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**Theorem.** Let K be a number field, and let  $\mathcal{D}_K$  be its ring of integers. Then  $\mathcal{D}_K$  is a Dedekind domain.

Structure theorem for finitely-generated  $\mathbb{Z}\text{-modules}.$ 

#### Structure theorem for finitely-generated $\mathbb{Z}$ -modules.

Let *M* be a finitely generated  $\mathbb{Z}$ -module. Then there exists a nonnegative integer *r* and a list (possibly empty) of integers  $n_1, \ldots, n_k$  with  $n_i > 1$  for all *i* such that *M* is isomorphic as a  $\mathbb{Z}$ -module to

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See the wiki page for the structure theorem for finitely generated modules over a PID.

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Idea: for  $0 \neq r \in R$ , consider the multiplication mapping

 $m_r\colon R\to R$  $s\mapsto rs.$ 

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