Math 361

February 20, 2023

Tomorrow's quiz

See posting at our homepage.

Note: This quiz will contain at least one question from previous weeks.

Today

Noetherian property

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- Equivalent conditions for the Noetherian property.

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- Equivalent conditions for the Noetherian property.
- Exact sequences of modules.
- Behavior of Noetherian property under mappings.
- Factorization in Noetherian domains.

Noetherian modules

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  So R is Noetherian if and only if all of its ideals are finitely generated.
- Every PID is a Noetherian ring. For instance, $\mathbb{Z}$ is Noetherian, and $K[x]$ is Noetherian when K is a field.

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For example, the ideal $I = (x_1, x_2, x_3 \ldots)$ is not finitely generated.

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3. Every nonempty collection of submodules of M has a maximal element under inclusion.

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Suppose M is Noetherian and let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of submodules. Let $N := \cup_{i \geq 1} N_i$.

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Since $N = \cup_{i \geq 1} N_i$, for each $i = 1, \ldots, s$, we have $n_i \in N_{k_i}$ for some k_i.

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Since $N = \cup_{i \geq 1} N_i$, for each $i = 1, \ldots, s$, we have $n_i \in N_{k_i}$ for some k_i. Let $k = \max\{k_i\}$.

Then $n_i \in N_k$ for all i. It follows that

$$N = \mathrm{Span}_R\{n_1, \ldots, n_s\} = N_k = N_{k+1} = \cdots .$$

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Pick $N_1 \in \mathcal{A}$. Is N_1 maximal in $\mathcal{A}$?

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Is N_2 maximal in $\mathcal{A}$?

If not, then there exists $N_3 \in \mathcal{A}$ such that $N_1 \subsetneq N_2 \subsetneq N_3$.

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If not, then there exists $N_3 \in \mathcal{A}$ such that $N_1 \subsetneq N_2 \subsetneq N_3$.

Repeat. Since every ascending chain eventually stabilizes, we must eventually reach a maximal element of $\mathcal{A}$.

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However, $N'' \in \mathcal{A}$, too, contradicting the maximality of N'. $\square$

Mappings of R-modules

A sequence of R-module mappings

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$M \xrightarrow{\psi} M'' \to 0$ is exact at M'' if and only if ψ is surjective.

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Identify M' with $\mathrm{im}(\phi) \subseteq M$. Then ψ induces an isomorphism

$$\begin{aligned} \overline{\psi}\colon M/M' &\to M'' \\ \overline{m} &\mapsto \psi(m). \end{aligned}$$

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Proof. $(\Rightarrow)$ Suppose M is Noetherian. We may assume $M' \subseteq M$. Is M' Noetherian? Every submodule of M' is a submodule of M, and hence is finitely generated. Therefore, M' is Noetherian.

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Next, suppose that N is a submodule of M''. Then $\psi^{-1}(N)$ is a submodule of M (exercise). Since M is Noetherian, $\psi^{-1}(N)$ is finitely generated, say by $n_1, \ldots, n_k$. Then $\psi(n_1), \ldots, \psi(n_k)$ generate N. Hence, M'' Noetherian.

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So $n \in \mathrm{Span}_R\{v_1, \ldots, v_\ell, n_1, \ldots, n_k\}$, as claimed. $\square$

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We have the short exact sequence

$$0 \to R \xrightarrow{\psi} R^n \xrightarrow{\phi} R^{n-1} \to 0.$$

where $\phi(r) = (r, 0, \ldots, 0)$ and $\psi(r_1, \ldots, r_n) = (r_2, \ldots, r_n)$.

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The result now follows by induction and the proposition we just proved. $\square$

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Proof. This follows since $\mathbb{Z}$ is a PID, hence, Noetherian, and $\mathfrak{O}_K$ is a finitely generated $\mathbb{Z}$-module. $\square$

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Does this contradict the Corollary?