Math 361

February 3, 2023

Quiz comments

Proposition. (Division algorithm) Let K be a field, and let $f, g \in K[x]$ with $f \neq 0$. Then there exists $q, r \in K[x]$ such that

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Definition. Let L/K be a field extension. Then $\alpha \in L$ is *algebraic* over K if there exists a nonzero polynomial $f \in K[x]$ such that $f(\alpha) = 0$.

Today

- Review.
- Gauss's lemma.
- Ring of integers in a number field.
- Extended primitive element theorem
- Ring of integers in a quadratic extension.

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Corollary. Let $A \subseteq B$ be domains. The set of elements of B that are integral over A forms a subring of B.

The algebraic integers

The *algebraic integers* are the elements of $\mathbb{C}$ that are integral over $\mathbb{Z}$:

$$\mathfrak{O} := \{\alpha \in \mathbb{C} : p(\alpha) = 0 \text{ for some monic } p \in \mathbb{Z}[x]\}.$$

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$$\lambda g, \ \frac{1}{\lambda} h \in \mathbb{Z}[x].$$

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Corollary. Let α be an algebraic number, i.e., a complex number that is algegraic over $\mathbb{Q}$. Then α is an algebraic integer if and only if its minimal polynomial over $\mathbb{Q}$ has integer coefficients.

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Therefore, $\lambda = \pm 1$. So $\frac{1}{\lambda} p = \pm p \in \mathbb{Z}[x]$, which implies $p \in \mathbb{Z}[x]$.$\square$

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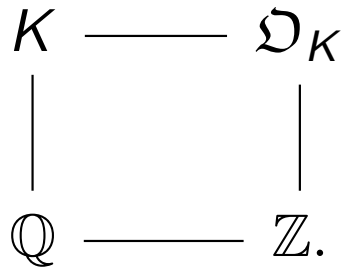
Extended primitive element theorem

$[K : \mathbb{Q}] < \infty$



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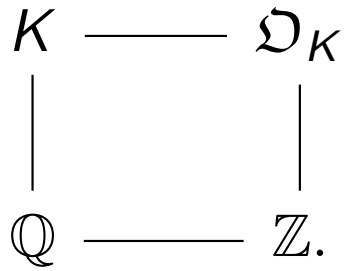

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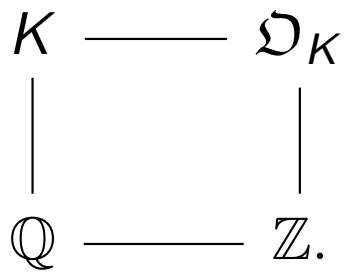

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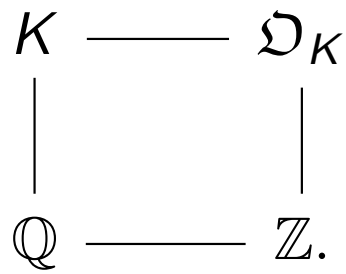

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Extended primitive element theorem. Let K be a number field. Then there exists $\theta \in \mathfrak{O}_K$ such that $K = \mathbb{Q}(\theta) = \mathbb{Q}[\theta]$.

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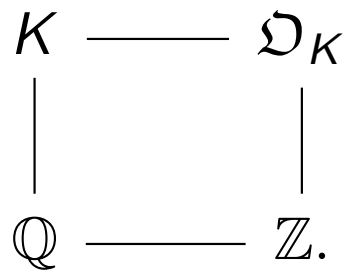

Lemma. If $\alpha \in K$, then there exists $c \in \mathbb{Z}$ such that $c\alpha \in \mathfrak{O}_K$.

Proof. HW.

Extended primitive element theorem. Let K be a number field. Then there exists $\theta \in \mathfrak{O}_K$ such that $K = \mathbb{Q}(\theta) = \mathbb{Q}[\theta]$.

Proof. The ordinary primitive element theorem gives us $\alpha \in K$ such that $K = \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$.

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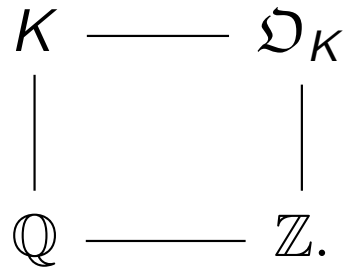

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Proof. The ordinary primitive element theorem gives us $\alpha \in K$ such that $K = \mathbb{Q}(\alpha) = \mathbb{Q}[\alpha]$. By the Lemma, there exists $c \in \mathbb{Z}$ such that $\theta := c\alpha \in \mathfrak{O}_K$.

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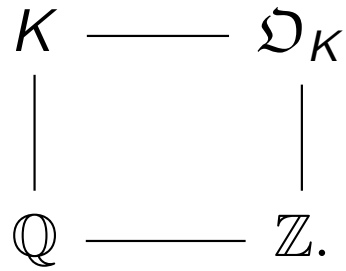

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Then $K = \mathbb{Q}(\alpha) = \mathbb{Q}(\theta)$ and $K = \mathbb{Q}[\alpha] = \mathbb{Q}[\theta]$. $\square$

Quadratic number fields

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Minimal polynomial for θ: $p = x^2 + mx + n \in \mathbb{Z}[x]$.

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So one possibility is $\alpha \in \mathbb{Z}$, but we already know $\mathbb{Z} \subset \mathfrak{O}_K$.

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From now on, suppose that $b \neq 0$.

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Thus, we know c is a power of 2.

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So $c = 1$ or 2.

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This shows that

$$\mathbb{Z}[\sqrt{d}] \subseteq \mathfrak{O}_K.$$

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That leaves the case $c = 2$ to consider.

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Since $a \in \mathbb{Z}$, the condition becomes $\frac{a^2 - b^2 d}{4} \in \mathbb{Z}$.

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In this case, $a^2 = b^2 = 1 \bmod 4$, and

$$a^2 - b^2 d = 1 - d = 0 \bmod 4.$$

So $d = 1 \bmod 4$.

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Theorem. Let K be a field extension of $\mathbb{Q}$ with $[K:\mathbb{Q}] = 2$. Then $K = \mathbb{Q}(\sqrt{d})$ where $d \neq 0, 1$ is a square-free integer. Its ring of integers is

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Example. If $K = \mathbb{Q}(\sqrt{5})$, then $\mathfrak{O}_K = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. On the other hand, $\mathfrak{O}_{\mathbb{Q}(\sqrt{7})} = \mathbb{Z}[\sqrt{7}]$.