Four lines problem

Plücker embedding of $\mathbb{G}_1\mathbb{P}^3$

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Taking coordinates $x(01), x(02), x(03), x(12), x(13), x(23)$ on $\mathbb{P}^5$, the image is exactly the set of solutions to the Plücker relation

$$x(01)x(23) - x(02)x(13) + x(03)x(12) = 0.$$

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So $\mathbb{G}_1\mathbb{P}^3$ is a *quadric hypersurface*: it has codimension 1 (dimension 4), and is defined by a single polynomial equation of degree 2.

Schubert variety: to meet a line

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The corresponding Schubert variety is

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Fact: under the Plücker embedding, $\mathbb{G}_1\mathbb{P}^3 \subset \mathbb{P}^5$, we have

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for some hyperplane H_L in $\mathbb{P}^5$ (consistent with the fact that $\mathbb{G}_1\mathbb{P}^3$ has codimension 1 in $\mathbb{P}^5$).

Meeting four lines

We have seen that the set of lines in $\mathbb{P}^3$ meeting a give line L is given by $\mathbb{G}_1\mathbb{P}^3 \cap H_L$ for some hyperplane $H_L \subset \mathbb{P}^5$.

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Goal: Compute a concrete example given four specific lines.

Generalized Laplace Expansion. Let D be an $n \times n$ matrix. Let $[n] = \{1, \ldots, n\}$, and fix row indices $I \subset [n]$. The complement is denoted $J = I^c = [n] \setminus I$. For each collection of column indices J having the same number of elements as I, i.e., $|I| = |J|$, we write $D_{I,J}$ for the corresponding submatrix of D. Then

$$\det D = \sum_{J \subset [n], |J| = |I|} (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} \det D_{I,J} \det D_{I^c, J^c}.$$

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Example.

$$D = \begin{pmatrix} 4 & 0 & 3 & 1 \\ 2 & 5 & 2 & 8 \\ 3 & 2 & 1 & 1 \\ 7 & 6 & 2 & 4 \end{pmatrix} \quad I = \{1, 2\} \quad J = \binom{[4]}{2}$$

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$$\det(D) = \cdots + \underbrace{(-1)^{(1+2)+(1+4)} \det \begin{pmatrix} 4 & 1 \\ 2 & 8 \end{pmatrix} \det \begin{pmatrix} 2 & 1 \\ 6 & 2 \end{pmatrix}}_{J = \{1,4\}} + \cdots$$

Condition: to meet a line

Consider lines ℓ and L in $\mathbb{P}^3$ with homogeneous coordinates in $\mathbb{G}_1\mathbb{P}^3$,

$$L = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix}, \qquad \ell = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$$

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These lines intersect if and only if

$$C = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$$

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has rank less than 4, i.e., if and only if $\det(C) = 0$.

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$$\mathfrak{S}(L, \mathbb{P}^3) = \mathbb{G}_1\mathbb{P}^3 \cap H_L.$$

Example

$$H_{L_1} = [23]x(01) - [13]x(02) + [12]x(03) + [03]x(12) - [02]x(13) + [01]x(23)$$

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Let (u, x, y, z) be the coordinates on $\mathbb{P}^3$, and consider the four lines given in homogeneous equations by

$$L_1 = \{y = z = 0\}, \qquad L_2 = \{x = z, y = u\}$$
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Spanning sets

$$L_1 \colon (1, 0, 0, 0), (0, 1, 0, 0)$$

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$$L_3\colon (0,2,0,1),(1,0,2,0) \quad L_4\colon (0,1,1,0),(1,0,0,1)$$

Corresponding condition hyperplanes, H_{L_i}:

$$H_{L_1} =$$

Example

$H_{L_1} = [23]x(01) - [13]x(02) + [12]x(03) + [03]x(12) - [02]x(13) + [01]x(23)$

Let (u, x, y, z) be the coordinates on $\mathbb{P}^3$, and consider the four lines given in homogeneous equations by

$$L_1 = \{y = z = 0\}, \qquad L_2 = \{x = z, y = u\}$$
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Example

A line $\ell \subset \mathbb{P}^3$ with Plücker coordinates $(x(01), \ldots, x(23)) \in \mathbb{P}^5$ meets lines $L_1, \ldots, L_4$ if and only if

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Solutions to this system of equations are the kernel of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0 & -1 \\ -2 & 0 & 4 & -1 & 0 & -2 \\ 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

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Basis for kernel: {(3,0,1,-2,-3,0),(0,1,0,0,1,0)}. So the line $\cap_{i=1}^{4} H_{L_i}$ has parametric equation

$$
\begin{aligned}
L(s, t) &= s(3, 0, 1, -2, -3, 0) + t(0, 1, 0, 0, 1, 0) \\
&= (3s, t, s, -2s, -3s + t, 0) \in \mathbb{P}^5.
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These are the Plücker coordinates for the two lines in $\mathbb{P}^3$ that meet $L_1,\ldots,L_4$.

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Plücker coordinates: $(3, 1, 1, -2, -2, 0)$ and $(3, 2, 1, -2, -1, 0)$.

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Since the second coordinate of $(3, 1, 1, -2, -2, 0)$ is 1, we solve

$$\wedge \begin{pmatrix} 1 & a & 0 & c \\ 0 & b & 1 & d \end{pmatrix} = (b, 1, d, a, ad - bc) = (3, 1, 1, -2, -2, 0).$$

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Thus, one line is given by

$$(s, t) \mapsto s(1, -2, 0, 0) + t(0, 3, 1, 1) \in \mathbb{P}^3,$$

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Thus, one line is given by

$$(s, t) \mapsto s(1, -2, 0, 0) + t(0, 3, 1, 1) \in \mathbb{P}^3,$$

or, in coordinates (u, x, y, z) for $\mathbb{P}^3$,

$$\{2u + x = 3z, y = z\}.$$

Example

Similarly,

$$\Lambda^2 \begin{pmatrix} 1 & a & c & 0 \\ 0 & b & d & 1 \end{pmatrix} = (b, d, 1, ad-bc, a, c) = (3, 2, 1, -2, -1, 0)$$

yields the other line:

$$(s, t) \mapsto s(1, -1, 0, 0) + t(0, 3, 2, 1),$$

or

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Embedding $\mathbb{R}^3 \subset \mathbb{P}^3$ as $\{u = 1\}$, we get lines in $\mathbb{R}^3$:

$$\{2u + x = 3z, y = z\} \rightsquigarrow \{2 + x = 3z, y = z\}$$

$$\{u + x = 3z, y = 2z\} \rightsquigarrow \{1 + x = 3z, y = 2z\}$$

Example

Two solutions in affine space:

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Our four lines in affine space:

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