PROBLEM 1. The Chow ring of $\mathbb{G}_1\mathbb{P}^4$.

(a) List all the Schubert classes for $\mathbb{G}_1\mathbb{P}^4$. For each class, state the codimension, give both the a-notation and λ-notation, and describe the associated Schubert condition. For example, one line of your list will be

codim	class	condition
3	$(1,3) = \{2,1\}$	to lie in a solid a meet a line lying in that solid

a "solid" being a 3-dimensional linear subspace.

(b) Make the 10×10 multiplication table for the Chow ring of $A^*(\mathbb{G}_1\mathbb{P}^4)$.

(c) Find m so that there will be a finite number of lines meeting m given lines, in general. Then find the number of lines meeting m general lines.

(d) How many lines will meet 6 planes, in general?

(e) What is the degree of $\mathbb{G}_1\mathbb{P}^4 \subset \mathbb{P}^9$ (Plücker embedding)? (Hint: The *degree* is the number of times a generic linear space of complementary dimension intersects the set. Since $\mathbb{G}_1\mathbb{P}^4$ has dimension $(r+1)(n-r) = 6$, i.e., codimension 3, the degree is given by the number of times a general solid (3-dimensional subspace, codimension 6) in $\mathbb{P}^9$ meets the Grassmannian. To create this solid, note that the Schubert class of codimension 1 is given by intersecting the Grassmannian with a hyperplane.)

(f) Create your own enumerative problem that can be answered using this Chow ring subject to the condition that the solution be a positive integer.

SOLUTION:

(a)

codim	class	condition
6	$(0,1) = \{3,3\}$	to be a given line
5	$(0,2) = \{3,2\}$	to lie in a plane and pass through a point in that plane
4	$(0,3) = \{3,1\}$	lie in a solid and pass through a given pt. in that solid
3	$(0,4) = \{3,0\}$	to pass through a point
4	$(1,2) = \{2,2\}$	to lie in a plane
3	$(1,3) = \{2,1\}$	to lie in a solid a meet a line lying in that solid
2	$(1,4) = \{2,0\}$	to meet a line
2	$(2,3) = \{1,1\}$	to lie in a solid
1	$(2,4) = \{1,0\}$	to meet a plane
0	$(3,4) = \{0,0\}$	no condition

(b)

	1	$\yng(1)$	$\yng(1,1)$	$\yng(2)$	$\yng(2,1)$	$\yng(3)$	$\yng(2,2)$	$\yng(3,1)$	$\yng(3,2)$	$\yng(3,3)$
1	1	$\yng(1)$	$\yng(1,1)$	$\yng(2)$	$\yng(2,1)$	$\yng(3)$	$\yng(2,2)$	$\yng(3,1)$	$\yng(3,2)$	$\yng(3,3)$
$\yng(1)$	$\yng(1)$	$\yng(1,1)+\yng(2)$	$\yng(2,1)$	$\yng(2,1)+\yng(3)$	$\yng(2,2)+\yng(3,1)$	$\yng(3,1)$	$\yng(3,2)$	$\yng(3,2)$	$\yng(3,3)$	0
$\yng(1,1)$	$\yng(1,1)$	$\yng(2,1)$	$\yng(2,2)$	$\yng(3,1)$	$\yng(3,2)$	0	$\yng(3,3)$	0	0	0
$\yng(2)$	$\yng(2)$	$\yng(2,1)+\yng(3)$	$\yng(3,1)$	$\yng(2,2)+\yng(3,1)$	$\yng(3,2)$	$\yng(3,2)$	0	$\yng(3,3)$	0	0
$\yng(2,1)$	$\yng(2,1)$	$\yng(2,2)+\yng(3,1)$	$\yng(3,2)$	$\yng(3,2)$	$\yng(3,3)$	0	0	0	0	0
$\yng(3)$	$\yng(3)$	$\yng(3,1)$	$\yng(3,2)$	$\yng(3,2)$	0	$\yng(3,3)$	0	0	0	0
$\yng(2,2)$	$\yng(2,2)$	$\yng(3,2)$	$\yng(3,3)$	0	0	0	0	0	0	0
$\yng(3,1)$	$\yng(3,1)$	$\yng(3,2)$	$\yng(3,3)$	$\yng(3,3)$	0	0	0	0	0	0
$\yng(3,2)$	$\yng(3,2)$	$\yng(3,3)$	0	0	0	0	0	0	0	0
$\yng(3,3)$	$\yng(3,3)$	0	0	0	0	0	0	0	0	0

(c) The condition of meeting a line is $(1,4) = \{2,0\}$, which has codimension 2. Since $\mathbb{G}_1\mathbb{P}^4$ has dimension 6 and codimensions add when multiplying classes, we have that there will be a finite number of lines meeting 3 generic lines ($6 = 3 \cdot 2$). To find the number of lines:

$$\begin{aligned}
\left(\yng(2)\right)^3 &= \yng(2) \cdot \yng(2)^2 \\
&= \yng(2) \cdot \left(\yng(2,2) + \yng(3,1)\right) \\
&= 0 + \yng(3,3) \\
&= \yng(3,3).
\end{aligned}$$

So there is 1 line meeting 3 generic lines.

(d) The condition of meeting a plane is $(2,4) = \{1,0\}$. Then

$$\begin{aligned}
\ydiagram{1}^6 &= \ydiagram{1}^4 \cdot \ydiagram{2}^2 \\
&= \ydiagram{1}^4 \cdot \left(\ydiagram{1,1} + \ydiagram{2} \right) \\
&= \ydiagram{1}^3 \cdot \left(2 \cdot \ydiagram{2,1} + \ydiagram{3} \right) \\
&= \ydiagram{1}^2 \cdot \left(3 \cdot \ydiagram{3,1} + 2 \cdot \ydiagram{2,2} \right) \\
&= \ydiagram{1} \cdot \left(5 \cdot \ydiagram{3,2} \right) \\
&= 5 \cdot \ydiagram{3,3}.
\end{aligned}$$

So there are 5 lines meeting 6 generic planes.

(e) A 3-dimensional linear subspace of $\mathbb{P}^9$ can be formed by intersecting 6 generic hyperplanes. The intersection of each of these hyperplanes with $\mathbb{G}_1\mathbb{P}^3$ gives the class $(2,4) = \{1,0\}$. So intersecting the 6 generic hyperplanes with $\mathbb{G}_1\mathbb{P}^4$ gives the class of $(2,4)^6$, which we calculated in the previous problem. So the degree is 5.

(f) How many lines meet two generic lines and meet two generic planes?

$$\begin{aligned}
\ydiagram{2}^2 \cdot \ydiagram{1}^2 &= \left(\ydiagram{2,2} + \ydiagram{3,1} \right) \cdot \ydiagram{1}^2 \\
&= \left(\ydiagram{3,2} + \ydiagram{3,2} \right) \cdot \ydiagram{1} \\
&= 2 \cdot \ydiagram{3,3}.
\end{aligned}$$

The answer is 2.

Problem 2. Give geometric explanations for the following calculations in $A^\bullet\mathbb{G}_1\mathbb{P}^3$:

(a) $(0,3)^2 = (1,2)^2 = (0,1)$;
(b) $(1,3)(0,3) = (1,3)(1,2) = (0,2)$;
(c) $(0,3)(1,2) = 0$;
(d) $(1,3)^2 = (0,3) + (1,2)$. (For this problem, use that fact that for some problems in Schubert calculus, it is OK to allow the conditions to degenerate. For instance, what if the two lines involved in $(1,3)^2$ were allowed to intersect? For a fuller explanation of this phenomenon, see Schubert calculus according to Schubert, by Felice Ronga.)

Solution:

(a) The class $(0,1)$ represents a single line. The relation $(0,3)^2 = (0,1)$ says that two distinct points determines a unique line. The relation $(1,2) = (0,1)$ says that two distinct planes in $\mathbb{P}^3$ determine a unique line (the intersection of the planes).

(b) The class $(0,2)$ represents a pencil of lines lying in a given plane (i.e., all line through a given point and lying in the plane). The class $(1,3)(0,3)$ are a lines passing through a fixed point and meeting a fixed line. These lines necessarily lie in the plane determined by the fixed point and line. The class $(1,3)(1,2)$ represents all line meeting a fixed line and lying in a fixed plane. Since the line and plane are in general position, they meet in a point. So all lines satisfying the condition must pass through this point.

(c) The condition $(0,3)$ represents all lines containing a fixed point, and the condition $(1,2)$ represents all lines lying in a fixed plane. Since the point and the plane are in generic position, the point will not lie in the plane. So there are no lines satisfying the condition.

(d) The condition $(1,3)^2$ represents all lines meeting two generic lines. Suppose that the lines are not generic, but meet at single point. Then the condition of meeting the two lines is met by lines either passing through the point of intersection or by lying in the plane determined by the two lines.