Schubert calculus

$\mathbb{G}_r\mathbb{P}^n$

Goal: Compute $H^k\mathbb{G}_r\mathbb{P}^n$.

Algebraic sets

An (affine) *algebraic set* X is the zero set of a set S of polynomial equations in $K[x_1, \ldots, x_n]$.

We can replace S with the ideal I generated by S without changing the solution set.

By the Hilbert basis theorem, I is finitely generated. So, $I = (f_1, \ldots, f_k)$ for some polynomials f_i. Therefore,

$$X = Z(I) = \{p \in K^n : f_1(p) = \cdots = f_k(p) = 0\}.$$

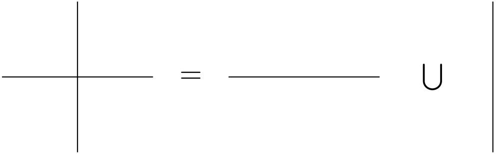
Example. $X = Z(z - x^2 - y^2, z - 4)$ is a circle of radius 2 in 3-space.

Varieties

An algebraic set $X = Z(I)$ is *irreducible* if it cannot be written as the proper union of two algebraic sets.

A *variety* is an irreducible algebraic set.

Example. $Z(xy)$ is the algebraic set $xy = 0$



$$= Z(y) \cup Z(x)$$

Projective varieties

A polynomial $f \in K[x_1, \ldots, x_{n+1}]$ is *homogeneous* if each of its monomials has the same degree.

Example: $f(x, y, z) = 3x^3 - 7xyz + 8yz^2$ is homogeneous of degree 3.

If f is homogeneous of degree d and $f(p) = 0$, then $f(\lambda p) = \lambda^d f(p) = 0$. Hence, f defines a subset of $\mathbb{P}^n$.

The solution set to a system of homogeneous polynomials is a *projective algebraic set*.

An projective algebraic set that cannot be written as a proper union of projective algebraic sets is *irreducible*.

An irreducible projective algebraic set is a *projective variety*.

Zariski topology

Exercise. The union of a finite number of algebraic sets is algebraic. The intersection of an arbitrary collection of algebraic sets is algebraic. The empty set and the whole space are algebraic (both for affine and projective algebraic sets).

The collection of algebraic sets form the closed sets of a topology called the *Zariski topology*.

Cycle space

$Z_r(X) =$ formal $\mathbb{Z}$-linear combinations of r-dimensional subvarieties of X

Rational equivalence. There is an algebraic geometry version of continuous deformation) $A \sim B$ for subvarieties $A, B \subseteq X$:

$A \sim B$ if there exists a subvariety $V \subset \mathbb{P}^1 \times X$ and points $p, q \in \mathbb{P}^1$ such that $V \cap (\{p\} \times X) - V \cap (\{q\} \times X) = A - B$.

Example. $Z(y - x^2) \sim Z(x^2)$ in $X = \mathbb{R}^2$ via $\alpha y - x^2$ for $\alpha \in \mathbb{R} \subset \mathbb{P}^1$.

Chow group

Chow group: $A^r(X) = Z_{n-r}(X)/\sim$

Chow ring: $A^\bullet(X) = \bigoplus_{r \geq 0} A^r(X)$

product: $[A][B] = [A \cap B]$

If $[A] \in A^r(X)$ and $[B] \in A^s(X)$, then $[A][B] \in A^{r+s}(X)$.

Chow ring of $\mathbb{G}_r\mathbb{P}^n$

flag: $r+1$ nested linear subspaces: $A_0 \subsetneq \cdots \subsetneq A_r \subseteq \mathbb{P}^n$

Schubert variety:

$$\mathfrak{S}(A_0, \dots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$$

Fact: Regarding $\mathbb{G}_r\mathbb{P}^n \subset \mathbb{P}^{\binom{n+1}{r+1}-1}$ via the Plücker embedding, there exists some linear subspace $M \subseteq \mathbb{P}^{\binom{n+1}{r+1}-1}$ such that

$$\mathfrak{S}(A_0, \dots, A_r) = \mathbb{G}_r\mathbb{P}^n \cap M.$$

Schubert classes

Notation for the rational equivalence class of a Schubert variety:

$$(a_0, \ldots, a_r) := [\mathfrak{S}(A_0, \ldots, A_r)] \in A^\bullet(\mathbb{G}_r\mathbb{P}^n)$$

where $a_i := \dim A_i$ for all i.

Theorem. If $A_0 \subsetneq \cdots \subsetneq A_r$ and $B_0 \subsetneq \cdots \subsetneq B_r$ are two flags with $a_i = \dim A_i$ and $b_i = \dim B_i$ for all i, then

$$(a_0, \ldots, a_r) = (b_0, \ldots, b_r) \in A^\bullet(\mathbb{G}_r\mathbb{P}^n)$$

Chow ring for $\mathbb{G}_r\mathbb{P}^n$

Theorem. The Chow ring for $\mathbb{G}_r\mathbb{P}^n$ is a free module with $\mathbb{Z}$-basis consisting of the Schubert classes:

$$\{(a_0, \ldots, a_r) : 0 \leq a_0 < \cdots < a_r \leq n\}.$$

The codimension of $(a_0, \ldots, a_r)$ is

$$\dim \mathbb{G}_r\mathbb{P}^n - \sum_{i=0}^{r}(a_i - 1) = (r+1)(n-r) - \sum_{i=1}^{r}(a_i - i).$$

A useful inequality

Lemma. Suppose L and M are linear subspaces of a vector space V of dimension n. Then

$$\dim(L \cap M) \geq \dim L + \dim M - n.$$

Proof.

Recall $L + M = \{\ell + m : \ell \in L \text{ and } m \in M\} \subseteq V$. There is a short exact sequence

$$0 \longrightarrow L \cap M \longrightarrow L \times M \longrightarrow L + M \longrightarrow 0$$

$$u \longmapsto (u, u)$$

$$(\ell, m) \longmapsto \ell - m$$

The result follows since the alternating sums of the dimensions of these space is 0. $\square$

A useful inequality

An r-plane L in $\mathbb{P}^n$ is a linear subspace of dimension $r+1$ in K^{n+1}. In the context of subspaces of projective spaces, we write $\dim L = r$. For the purposes of the following corollary, we will write $\mathrm{pdim}(L) = r$. So $\mathrm{pdim}(L) = \dim(L) - 1$.

Corollary. Suppose L and M are r- and s-planes in $\mathbb{P}^n$, respectively. Then

$$\mathrm{pdim}(L \cap M) \geq \mathrm{pdim}(L) + \mathrm{pdim}(M) - n.$$

Proof.

$$\begin{aligned}
\mathrm{pdim}(L \cap M) &= \dim(L \cap M) - 1 \\
&\geq \dim(L) + \dim(M) - (n+1) - 1 \\
&= (\mathrm{dim}(L) - 1) + (\mathrm{dim}(M) - 1) - n \\
&= \mathrm{pdim}(L) + \mathrm{pdim}(M) - n.
\end{aligned}$$

□

Examples of Schubert classes

Recall:

$$\mathfrak{S}(A_0, \ldots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$$

and $(a_0, \ldots, a_r)$ is the corresponding class in $\mathbb{G}_r\mathbb{P}^n$

Consider $(2, 3) \in A^\bullet(\mathbb{G}_1\mathbb{P}^3)$.

Flag: a plane sitting in 3-space, $\mathbb{P}^3$.

$A_0 =$ 2-plane in 3-space, and $A_1 =$ 3-plane in 3-space.

$\dim(L \cap A_0) \geq 0$ is satisfied by all L (no condition on L).

$\dim(L \cap A_1) \geq 1$ is also no condition.

$(2, 3) = [\mathbb{G}_1\mathbb{P}^3]$, no condition. This class is the multiplicative identity in the Chow (intersection) ring.

Examples

$(2,3) = [\mathbb{G}_1\mathbb{P}^3]$

What is the codimension of $(2,3)$?

The *dimension* is

$$\sum_{i=0}^{r}(a_i - i) = \sum_{i=0}^{1}(a_i - i) = (2-0) + (3-1) = 4.$$

So the codimension is

$$\dim \mathbb{G}_1\mathbb{P}^3 - 4 = \dim G(2,4) - 4 = 2(4-2) - 4 = 0,$$

as expected.

Examples

$$\mathfrak{S}(A_0, \ldots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$$

and $(a_0, \ldots, a_r)$ is the corresponding class in $\mathbb{G}_r\mathbb{P}^n$

Now consider $(0, 2) \in A^\bullet\mathbb{G}_1\mathbb{P}^3$.

Flag: a point sitting in a plane.

$\dim(L \cap A_0) \geq 0$ iff $A_0 \in L$.

$\dim(L \cap A_1) \geq 1$ iff $L \subset A_1$.

So $(0, 2) \in \mathbb{G}_1\mathbb{P}^3$ represents the set of lines sitting in a given plane and passing through a given point in that plane.

Condition: To sit in a given plane and pass through a given point.

Examples

The dimension of $(0,2)$ is

$$\sum_{i=0}^{1}(a_i - i) = (0-0) + (2-1) = 1.$$

So the codimension of $(0,2)$ is

$$\dim \mathbb{G}_1\mathbb{P}^3 - 1 = 4 - 1 = 3.$$

Why does this make geometric sense?

Examples

$(0, 3, 4) \in A^\bullet \mathbb{G}_2\mathbb{P}^4$

Conditions on $L \in \mathbb{G}_2\mathbb{P}^4$:

$\dim(L \cap A_0) \geq 0$: L passes through a given point A_0

$\dim(L \cap A_1) \geq 1$: no condition

$\dim(L \cap A_2) \geq 2$: no condition

So $(0, 3, 4)$ is the class of planes passing through a given point.

Codimension:

$$\dim \mathbb{G}_2\mathbb{P}^4 - \sum_{i=0}^{2}(a_i - i) = 6 - (0 - 0) - (3 - 1) - (4 - 2) = 2.$$