Schubert calculus II

Young diagrams

$\mathbb{G}_r\mathbb{P}^n \qquad 0 \subseteq A_0 \subsetneq \cdots \subsetneq A_r \subseteq \mathbb{P}^n \qquad 0 \leq a_0 < \cdots < a_r \leq n$

Schubert variety: $\{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}$

Schubert class:

$$(a_0, \ldots, a_r) = [\{L \in \mathbb{G}_r\mathbb{P}^n : \dim(L \cap A_i) \geq i \text{ for all } i\}] \subseteq A^k\mathbb{G}_r\mathbb{P}^n$$

Codimension:

$$\begin{aligned} k &= \operatorname{codim}(a_0, \ldots, a_r) \\ &= \dim \mathbb{G}_r\mathbb{P}^n - \sum_{i=0}^{r}(a_i - i) \\ &= (r+1)(n-r) - \sum_{i=0}^{r}(a_i - i) \\ &= \sum_{i=0}^{r}(\underbrace{(n-r) - (a_i - i)}_{\lambda_i}). \end{aligned}$$

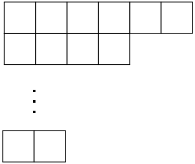
Young diagrams

$$\operatorname{codim}(a_0, \dots, a_r) = \sum_{i=0}^{r} \underbrace{((n-r) - (a_i - i))}_{\lambda_i}$$

$$n - r \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$$

$$|\lambda| := \lambda_0 + \lambda_1 + \cdots + \lambda_r = \operatorname{codim}(a_0, \dots, a_r)$$

Young diagram:



$\leftarrow \lambda_0$ boxes

$\leftarrow \lambda_1$ boxes

$\vdots$

$\leftarrow \lambda_r$ boxes

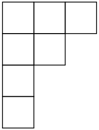
$$\{\lambda\} = (a_0, \dots, a_r) \in A^{|\lambda|}\mathbb{G}_r\mathbb{P}^n$$

Young diagram: Example

$a = (0, 2, 4, 5, 7) \in A^7\mathbb{G}_4\mathbb{P}^7$

$a_i - i$: $\quad a - (0, 1, 2, 3, 4) = (0, 1, 2, 2, 3)$

$\lambda_i = (n - r) - (a_i - i) = 3 - (a_i - i)$: $\quad \lambda = (3, 2, 1, 1, 0)$



$$\{\lambda\} = \{3, 2, 1, 1, 0\} \in A^7\mathbb{G}_4\mathbb{P}^7$$

Ring structure for $A^\bullet \mathbb{G}_r \mathbb{P}^n$

$$\{\lambda\} \cdot \{\mu\} = \sum_{\nu \in A^\bullet \mathbb{G}_r \mathbb{P}^n} c_{\lambda\mu}^{\nu} \{\nu\}$$

$c_{\lambda\mu}^{\nu}$ are the *Littlewood-Richardson numbers*.

- #P-complete
- May be computed using Knutson-Tao puzzles (see Video of the Week).
- We will use the following definition:

$c_{\lambda\mu}^{\nu} =$ the number of strict μ-expansions of λ resulting in ν.

Ring structure for $A^\bullet \mathbb{G}_r \mathbb{P}^n$

$$\{\lambda\} \cdot \{\mu\} = \sum_{\nu \in A^\bullet \mathbb{G}_r \mathbb{P}^n} c^{\nu}_{\lambda\mu} \{\nu\}$$

$c^{\nu}_{\lambda\mu} =$ the number of strict μ-expansions of λ resulting in ν.

- The i-th row of the YD for μ has μ_i boxes. Label each of these with the number i.
- Start with λ. At the i-th step of the expansion, add the boxes from row i of μ to λ subject to the following rules:
  - no two can be in the same column
  - the resulting shape must be a YD
  - at the end, reading right-to-left, top-to-bottom, there can never be more $(i+1)$s than is.
  - at the end, the resulting YD must fit in an $(r+1) \times (n-r)$ rectangle.

See Example 14.31 in text.

Ring structure: Example

In $\mathbb{G}_3\mathbb{P}^8$,

$$\begin{aligned}\{4,3,2,0\}\cdot\{4,2,1,0\} = \{4,4,4,4\} &+ 3\{5,4,4,3\} + 2\{5,5,3,3\} \\ &+ 3\{5,5,4,2\} + \{5,5,5,1\}.\end{aligned}$$

Check some of these (on board).

See Examples 14.28 and 14.34 in text.