Grassmannians: Plücker coordinates

Plücker embedding

$W \in G(r, V)$

$\dim \Lambda^r W = 1$

W : basis $w_1, \ldots, w_r \quad \Rightarrow \quad \Lambda^r W = \mathrm{Span}\{w_1 \wedge \cdots \wedge w_r\}$

$W \subseteq V \quad \Rightarrow \quad \Lambda^r W \subseteq \Lambda^r V$

Plücker embedding:

$$\begin{gathered} G(r, V) \to \mathbb{P}(\Lambda^r V) \\ W \mapsto \Lambda^r W \end{gathered}$$

Plücker embedding: coordinates

$V = \mathbb{R}^n$

basis for $\mathbb{R}^n$: $\{e_{j_1} \wedge \cdots \wedge e_{j_r} : j : 1 \leq j_1 < \cdots < j_r \leq n\}$

$$W : \begin{pmatrix} \text{---} & w_1 & \text{---} \\ & \vdots & \\ \text{---} & w_r & \text{---} \end{pmatrix}$$

$w_1 \wedge \cdots \wedge w_r = \sum_j \det(W_j) e_{j_1} \wedge \cdots \wedge e_{j_r}$

Plücker embedding:

$$\Lambda\colon\ G(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$$
$$r \times n \text{ matrix } L \mapsto \Lambda L = (\det(L_j) : j\colon 1 \leq j_1 < \cdots < j_r \leq n)$$

Plücker embedding: coordinates

Plücker embedding:

$$\Lambda\colon\ G(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$$

$$r \times n \text{ matrix } L \mapsto \Lambda L = (\det(L_j) : j\colon 1 \le j_1 < \cdots < j_r \le n)$$

Example.

$$\Lambda\colon\ G(2,4) \to \mathbb{P}^5$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix} \mapsto (-2, 1, -7, 2, -8, -3)$$

$\dim G(2,4) = 4$

So $G(2,4)$ is embedded as a *hypersurface* in $\mathbb{P}^5$.

Plücker relations

$\Lambda\colon G(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$

Coordinates on $\mathbb{P}^{\binom{n}{r}-1}$: $x(j) = x(j_1, \ldots, j_r)$ for each $j : 1 \le j_1 < \cdots < j_r \le n$.

For convenience, we define $x(j)$ for all $j \in \{1, \ldots, n\}^r$ with the following rules:

1. For each permutation $\sigma \in \mathfrak{S}_n$, set $x(\sigma(j)) = \mathrm{sign}(\sigma) x(j)$
2. $x(j) = 0$ if $j_s = j_t$ for some $s \neq t$.

Examples: $x(1,3,5) = -x(3,1,5) = x(5,1,3)$, and $x(5,2,2) = 0$

Plücker relations

$I\colon 1 \leq i_1 < \cdots < i_{r-1} \leq n$

$J\colon 1 \leq j_1 < \cdots < j_{r+1} \leq n$

$P_{I,J} = \sum_{k=1}^{r+1} (-1)^k x(i_1, \ldots, i_{r-1}, j_k) x(j_1, \ldots, \widehat{j_k}, \ldots, j_{r+1})$

Proposition.

$$\mathrm{im}(\Lambda) = \{x \in \mathbb{P}^{\binom{n}{r}-1} : P_{I,J}(x) = 0 \text{ for all } I, J\}$$

The image of $G(r, n)$ under the Plücker embedding is the zero set of the ideal generated by the Plücker relations.

Plücker relations

Example. $G(2,4)$, $\quad I = \{1\},\ J = \{2,3,4\}$

$$P_{1,234} = -x(12)x(34) + x(13)x(24) - x(14)x(23)$$

$$P_{2,134} = -x(21)x(34) + x(23)x(14) - x(24)x(13)$$

Note: $P_{1,234} = -P_{2,134}$. So $P_{1,234}(q) = 0 \Leftrightarrow P_{2,134}(q) = 0$.

In fact, the image of $G(2,4)$ under the Plücker embedding is defined just $P_{1,234}$.

Exercise. Compute the Plücker coordinates for $W = \mathrm{Span}\{(1,1,0,1),(0,2,0,1)\} \subset \mathbb{R}^4$ in $G(2,4)$, and check they satisfy $P_{1,234}$.

Proof that im(Λ) satisfies all Plücker relations

$$P_{I,J}(\Lambda(L))$$

$$= \sum_{k=1}^{r+1} (-1)^k \det L_{i_1,\ldots,i_{r-1},j_k} \det L_{j_1,\ldots,\widehat{j_k},\ldots,j_{r+1}}$$

$$= \sum_{k=1}^{r+1} (-1)^k \underbrace{\begin{vmatrix} \ldots & \begin{matrix} a_{1j_k} \\ \vdots \\ a_{r,j_k} \end{matrix} \end{vmatrix}}_{\text{expand}} \begin{vmatrix} \ldots & \begin{matrix} \widehat{a_{1j_k}} \\ \vdots \\ \widehat{a_{rj_k}} \end{matrix} & \ldots \end{vmatrix}$$

$$= \pm \sum_{k=1}^{r+1} (-1)^k \sum_{\ell=1}^{r} (-1)^\ell a_{\ell j_k} \begin{vmatrix} & \vdots & \\ \widehat{a_{\ell i_1}} & \cdots & \widehat{a_{\ell i_{r-1}}} \\ & \vdots & \end{vmatrix} \begin{vmatrix} \ldots & \begin{matrix} \widehat{a_{1j_k}} \\ \vdots \\ \widehat{a_{rj_k}} \end{matrix} & \ldots \end{vmatrix}$$

$$= \pm \sum_{\ell=1}^{r} (-1)^\ell \begin{vmatrix} & \vdots & \\ \widehat{a_{\ell i_1}} & \cdots & \widehat{a_{\ell i_{r-1}}} \\ & \vdots & \end{vmatrix} \left(\sum_{k=1}^{r+1} (-1)^k a_{\ell j_k} \begin{vmatrix} \ldots & \begin{matrix} \widehat{a_{1j_k}} \\ \vdots \\ \widehat{a_{rj_k}} \end{matrix} & \ldots \end{vmatrix} \right)$$

Plücker relations

$$
\begin{aligned}
P_{I,J}(\Lambda(L)) &= \cdots \\
&= \pm \sum_{\ell=1}^{r} (-1)^{\ell} \begin{vmatrix} & \vdots & \\ \widehat{a_{\ell i_1}} & \cdots & \widehat{a_{\ell i_{r-1}}} \\ & \vdots & \end{vmatrix} \left(\sum_{k=1}^{r+1} (-1)^k a_{\ell j_k} \begin{vmatrix} & \widehat{a_{1 j_k}} & \\ \ldots & \vdots & \ldots \\ & \widehat{a_{r j_k}} & \end{vmatrix} \right) \\
&= \pm \sum_{\ell=1}^{r} (-1)^{\ell} \begin{vmatrix} & \vdots & \\ \widehat{a_{\ell i_1}} & \cdots & \widehat{a_{\ell i_{r-1}}} \\ & \vdots & \end{vmatrix} \left(\begin{vmatrix} a_{\ell j_1} & \cdots & a_{\ell j_k} & \cdots & a_{\ell j_{r+1}} \\ a_{1 j_1} & \cdots & a_{1 j_k} & \cdots & a_{1 j_{r+1}} \\ & & \vdots & & \\ a_{r j_1} & \cdots & a_{r j_k} & \cdots & a_{r j_{r+1}} \end{vmatrix} \right) \\
&= 0.
\end{aligned}
$$