Grassmannians: manifold structure

Homework

Homework due Wednesday.

Question motivated by last week's HW

What is wrong with the following argument?

Let M be a compact n-manifold with boundary N, and let $[\omega] \in H^{n-1}N$. Then, applying Stokes' theorem,

$$\int_N \omega = \int_{\partial M} \omega = \int_M d\omega = \int_M 0 = 0.$$

Notation for the Grassmannian

Let V be a finite-dimensional vector space over a field K. The set of r-dimensional subspaces of V is the *Grassmannian*, $G(r, V)$. We write $G(r, n) := G(r, K^n)$.

Projective space over V:

$$\mathbb{P}(V) := G(1, V).$$

Grassmannian of r-planes in $\mathbb{P}(V)$:

$$\mathbb{G}_r\mathbb{P}^n := G(r+1, n+1).$$

Example on board: Why think of $\mathbb{G}_1\mathbb{P}^2$ as planes in 3-space?

Homogeneous coordinates for $G(r, n)$

$W \subseteq \mathbb{R}^n$ an r-dimensional subspace

Homogeneous coordinates for $W \in G(r, n)$ are given by any $r \times n$ matrix L with rowspace equal to W.

For instance, if $W = \text{Span}\{v_1, \ldots, v_r\}$, then we get the homogeneous coordinates

$$L = \begin{pmatrix} \text{---} & v_1 & \text{---} \\ & \vdots & \\ \text{---} & v_r & \text{---} \end{pmatrix}$$

Homogeneous coordinates for $G(r, n)$

Change of basis for W:

$$L = \begin{pmatrix} \text{---} & v_1 & \text{---} \\ & \vdots & \\ \text{---} & v_r & \text{---} \end{pmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{pmatrix} \text{---} & \tilde{v}_1 & \text{---} \\ & \vdots & \\ \text{---} & \tilde{v}_r & \text{---} \end{pmatrix}$$

i.e.

$$ML = M \begin{pmatrix} \text{---} & v_1 & \text{---} \\ & \vdots & \\ \text{---} & v_r & \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} & \tilde{v}_1 & \text{---} \\ & \vdots & \\ \text{---} & \tilde{v}_r & \text{---} \end{pmatrix}$$

Alternative definition:

$$G(r, n) = \{r \times n \text{ matrices of rank } r\} \Big/ \{L \sim ML : M \in \mathrm{GL}(r)\}.$$

Homogeneous coordinates for $G(r, n)$

Special case: $\mathbb{P}^n = G(1, n+1) = \mathbb{G}_0\mathbb{P}^n$:

$$\mathbb{P}^n = \{1 \times (n+1) \text{ matrices of rank } 1\} \Big/ \{L \sim ML : M \in \mathrm{GL}(1)\}.$$

Manifold structure for $G(r, n)$

We will find coordinates of the following point in $G(2,4)$ with respect to columns $j = (1,4)$:

$$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Row reduce so that columns indexed by j become the identity I_2:

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 4 & -3 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -4 & 3 & 1 \end{pmatrix}$$

Flatten, skipping entries in columns j, to get a point in $\mathbb{R}^4$ corresponding to W

$$\begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & -4 & 3 & 1 \end{pmatrix} \xrightarrow{\sim} (4, 0, -4, 3) \in \mathbb{R}^4.$$

Manifold structure for $G(r, n)$

$$W = \mathrm{Span}\{v_1, \ldots, v_r\}, \quad L = \begin{pmatrix} \text{---} & v_1 & \text{---} \\ & \vdots & \\ \text{---} & v_r & \text{---} \end{pmatrix}$$

Since L has rank r, some set of r columns of L is independent.

Say their column indices are $j = (j_1, \ldots, j_r)$. Let L_j be the submatrix of L consisting of these columns.

Then $\tilde{L} := L_j^{-1} L$ has the same rowspan as L and the submatrix $\tilde{L}$ consisting of columns j is the identity matrix I_r.

Define $\mathrm{flatten}_j(L_j^{-1} L) \in \mathbb{R}^{r(n-r)}$ to be the entries of $L_j^{-1} L$, read from along rows, top-to-bottom, skipping entries in columns $j_1, \ldots, j_r$.

Manifold structure for $G(r, n)$

Atlas: $\{U_{j,\phi_j}\}_j \qquad j : 1 \leq j_1 < j_2 < \cdots < j_r \leq n$

$U_j = \{r \times n \text{ matrices } L : \; L_j \text{ has rank } r\}/L \sim ML$

$$\begin{aligned} \phi_j \colon U_j &\to \mathbb{R}^{r(n-r)} \\ L &\mapsto \mathrm{flatten}_j(L_j^{-1}L) \end{aligned}$$

Why is ϕ_j a homeomorphism?

- U_j has the quotient topology.
- $L \mapsto L_j^{-1}$ is smooth (cf. adjugate formula for inverses).
- $L \mapsto L_j^{-1}L$ is smooth (entries are polynomials in entries. of L)
- Dropping columns of a matrix is smooth.

Manifold structure for $G(r, n)$

Exercise. Find the coordinates of the point in $G(r, n)$ with homogeneous coordinates

$$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in U_{(1,3)}$$

with respect to $j = (1, 3)$.

Translating from affine to projective geometry

The following point in $G(2,4) = \mathbb{G}_1\mathbb{P}^3$ can be thought of as the line in $\mathbb{R}^3$ through the two points $(1,0,3)$ and $(2,4,3)$:

$$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Reason: think of $\mathbb{R}^3$ as $U_3 = \{(x,y,z,w) \in \mathbb{P}^3 : w \neq 0\}$:

$\mathbb{G}_1\mathbb{P}^3$

A point in $\mathbb{G}_1\mathbb{P}^3$ represents a line in 3-space.

A line in $\mathbb{R}^3$ is determined by choosing a point $p = (x, y, z)$ and a direction $u = (u_1, u_2, u_3)$.

So p and u are both coming from 3-space. What should $\dim \mathbb{G}_1\mathbb{P}^3$ be?

Duality

An $(n-1)$-plane in $\mathbb{P}^n$ is called a *hyperplane*. It is the solution set to a linear equation

$$H_a := (a_1, \ldots, a_{n+1}) \cdot (x_1, \ldots, x_{n+1}) = a_1 x_1 + \cdots + a_{n+1} x_{n+1} = 0$$

for some $a := (a_1, \ldots, a_{n+1}) \neq (0, \ldots, 0) \in K^{n+1}$.

The hyperplane H_a is determined up to $a \sim \lambda a$ for $\lambda \in K \setminus \{0\}$. So we may regard

Define the *dual projective space* $(\mathbb{P}^n)^* = \mathbb{G}_{n-1}\mathbb{P}^n$.

We have a bijection:

$$\mathbb{P}^n \to (\mathbb{P}^n)^*$$
$$a \mapsto H_a.$$

Generalization: $G(k, n) \xrightarrow{\sim} G(n-k, n)$.