Toric Varieties: polytopes, cohomology

Quiz

- What does it mean to say $f\colon M \to N$ is *null-homotopic*, and what can we then conclude about the cohomology of M and N?
- What does it mean to say M is *contractible*, and what can we then conclude about the homology of M? Give a concrete example of a contractible space.
- Give, with proof, an example of a manifold that is not contractible.

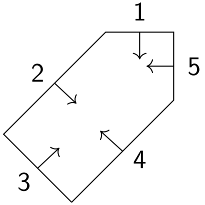
Geometry $\leftrightarrow$ Combinatorics

1. Compactness: A TV is compact iff its fan covers $N_{\mathbb{R}}$ (we say the fan is *complete*).
2. Smoothness: A TV is smooth iff $\sigma \cap N$ is generated by part of a $\mathbb{Z}$-basis for N for each cone σ.
3. Orientability: A smooth TV is symplectic, hence orientable. (Warning example: $\mathbb{P}^2_{\mathbb{C}}$ versus $\mathbb{P}^2_{\mathbb{R}}$.)

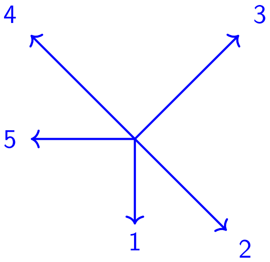
Polytopes $\to$ toric varieties

Let $P \subset M \approx \mathbb{R}^n$ be a rational polytope (the smallest convex set containing a finite set of points with rational coordinates).

Then the inward pointing normals to the facets of P determine a complete fan, $\Delta(P)$.



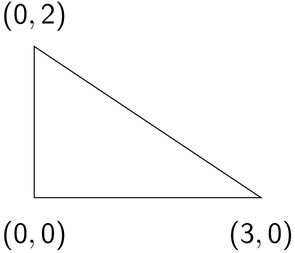
$P \subset M_{\mathbb{R}}$



$\Delta(P) \subset N_{\mathbb{R}}$

Polytopes → toric varieties

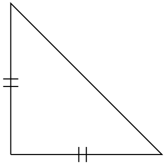
Example. We saw last time that the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ has corresponding toric variety $\mathbb{P}^2$. What about the following triangle:



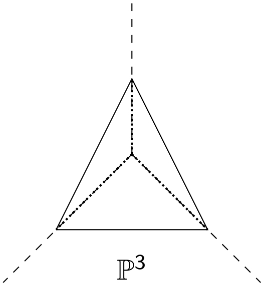
(Example to be done on board.)

Polytopes $\rightarrow$ toric varieties: more examples

$\mathbb{P}^1$

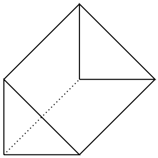


$\mathbb{P}^2$



$\mathbb{P}^3$

$\mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{P}^2 \times \mathbb{P}^1$

Chow ring & cohomology

Δ complete fan, $X = X(\Delta)$ corresponding TV, assumed smooth.

$\Delta(1) = \{D_1, \ldots, D_k\}$ = one-dimensional cones (rays) of Δ

$\mathbb{Z}[D_1, \ldots, D_k]$ polynomial ring (treating D_i as indeterminates)

Chow ring: $A^\bullet X := \oplus_{k \geq 0} A^k(X) = \mathbb{Z}[D_1, \ldots, D_k]/(I + J)$

$I = \{\prod_{D \in S} D : S \subset \Delta(1)$ and S does not span a cone$\}$

$J = \{\sum_{D \in \Delta(1)} \langle e_i, n_D \rangle D : i = 1, \ldots, n\}$ where n_D is the first lattice point along D (and e_i is the i-th basis vector for N

Cohomology: $H^k X = A^k X \otimes_{\mathbb{Z}} \mathbb{R}$ if k is even, and $H^k X = 0$ if k is odd

Example: $\mathbb{P}^2$, on the board