Toric Varieties: Hirzebruch surfaces

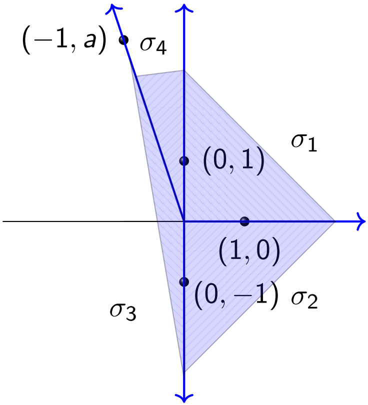
Homework

Homework for next week.

Review

See the cohomology section of last Wednesday's lecture.

Hirzebruch surfaces



$a \in \mathbb{Z}_{>0}$

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[u, v], \quad U_{\sigma_1} \approx \mathbb{C}^2; \quad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[x, y], \quad U_{\sigma_2} \approx \mathbb{C}^2,$$

How does U_{σ_1} glue into U_{σ_4} and U_{σ_2} glue into U_{σ_3}?

$\mathbb{P}^1$ bundle

There exists a mapping $\pi\colon H_a \to \mathbb{P}^1$ such that $\pi^{-1}(p) \approx \mathbb{P}^1$ for all $p \in \mathbb{P}^1$. So H_a is *a* $\mathbb{P}^1$ *bundle over* $\mathbb{P}^1$.

Underlying reason: What does the projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ do to the lattice N and the cones? (See our notes.)

$\mathbb{P}^1$ bundle

Fill in the following diagram:

$$
\begin{tikzcd}
U_{\sigma_4} & (\quad , \quad) \arrow[r, leftrightarrow, dashed] \arrow[d, leftrightarrow, dashed] & (u, v) \arrow[d, leftrightarrow, dashed] & U_{\sigma_1} \\
U_{\sigma_3} \arrow[d] & (\quad , \quad) \arrow[r, leftrightarrow, dashed] \arrow[d, mapsto] & (\quad , \quad) \arrow[d, mapsto] & U_{\sigma_2} \arrow[d] \\
U_{\tau_2} & \quad \arrow[r, leftrightarrow, dashed] & u & U_{\tau_1}
\end{tikzcd}
$$

$\mathbb{P}^1$ bundle

Solution: Fill in the following diagram:

$$
\begin{tikzcd}
U_{\sigma_4} & (1/u, u^a v) \arrow[r, leftrightarrow, dashed] \arrow[d, leftrightarrow, dashed] & (u, v) \arrow[d, leftrightarrow, dashed] & U_{\sigma_1} \\
U_{\sigma_3} \arrow[d] & (1/(u), 1/(u^a v)) \arrow[r, leftrightarrow, dashed] \arrow[d, mapsto] & (u, 1/v) \arrow[d, mapsto] & U_{\sigma_2} \arrow[d] \\
U_{\tau_2} & 1/u \arrow[r, leftrightarrow, dashed] & u & U_{\tau_1}
\end{tikzcd}
$$

Generalization

Let Δ be a fan for lattice N and Δ' a fan for lattice N'.

Suppose there exist a $\mathbb{Z}$-linear mapping $\phi\colon N \to N'$ such that for each $\sigma \in \Delta$, there is a cone $\sigma' \in \Delta'$ with $\phi(\sigma) \subseteq \sigma'$.

Then there is a natural mapping $X(\Delta) \to X(\Delta')$ induced by ϕ.

Cohomology of the Hirzebruch surface

Problem: compute $H^k(H_a)$. (Solution: See Example 13.14 in our notes.)

Recall: Δ complete fan, $X = X(\Delta)$ corresponding TV, assumed smooth.

$\Delta(1) = \{D_1, \ldots, D_k\}$ = one-dimensional cones (rays) of Δ

$\mathbb{Z}[D_1, \ldots, D_k]$ polynomial ring (treating D_i as indeterminates)

Chow ring: $A^\bullet X := \oplus_{k \geq 0} A^k(X) = \mathbb{Z}[D_1, \ldots, D_k]/(I + J)$

$I = \{\prod_{D \in S} D : S \subset \Delta(1) \text{ and } S \text{ does not span a cone}\}$

$J = \{\sum_{D \in \Delta(1)} \langle e_i, n_D \rangle D : i = 1, \ldots, n\}$ where n_D is the first lattice point along D (and e_i is the i-th basis vector for N

Cohomology: $H^k X = A^k X \otimes_{\mathbb{Z}} \mathbb{R}$ if k is even, and $H^k X = 0$ if k is odd