More de Rham cohomology results

Quiz

1. What is the k-th de Rham cohomology group of a manifold M?
2. Let $f, g\colon M \to N$ be two mappings between manifolds.
   (i) What does it mean to say f is *homotopic* to g?
   (ii) State the homotopy invariance theorem for de Rham cohomology.

Locally constant functions

Let X be a topological space. Then X is *connected* if it is not the union of two disjoint nonempty open sets. Equivalently, the is no proper subset of X that is both open and closed.

A function $f: X \to \mathbb{R}$ is *locally constant* if for all $p \in X$, there exists a neighborhood U of p such that $f|_U$ is constant.

Suppose that $f: X \to \mathbb{R}$ is continuous and locally constant. What can we say?

Proposition. If $f: X \to \mathbb{R}$ is a continuous locally constant function and X is connected, then f is constant.

Proof?

Locally constant functions

Proposition. If $f: X \to \mathbb{R}$ is a continuous locally constant function and X is connected, then f is constant.

Proof. Let $p \in X$ and suppose that $f(p) = a \in \mathbb{R}$. Let

$$U = \{x \in X : f(x) = a\}.$$

Then U is nonempty since $p \in U$. The set U is also open: given $q \in U$, since f is locally constant, there exists a neighborhood W of q such that $f|_W$ is constant. It follows that $W \subseteq U$ is an open neighborhood of q contained in U.

Since $\{a\} \subset \mathbb{R}$ is a closed set and f is continuous, $f^{-1}(a)$ is closed. Therefore, $X \setminus U = X \setminus f^{-1}(a)$ is open.

Since U and $X \setminus U$ are open and X is connected, it follows that $U = X$, which means f is constant. $\square$

Null-homotopic

A mapping of manifolds $f\colon M \to N$ is *null-homotopic* if it is homotopic to a constant mapping.

Proposition. If $f\colon M \to N$ is null-homotopic, then

$$f^{*,k}\colon H^k(N) \to H^k(M)$$

is the zero mapping for all $k > 0$.

Proof. Suppose that $f \sim g$ where $g\colon M \to N$ is constant. By the homotopy invariance theorem, $f^{*,k} = g^{*,k}$ for $k \geq 0$. The result follows since $g^{*,k} = 0$ for all $k > 0$. □

Contractible manifolds

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is the zero mapping for all $k > 0$.

A manifold M is *contractible* if id_M is null-homotopic. Examples?

Proposition. Suppose M is contractible. Then $H^k(M) = 0$ for all $k > 0$.

Proof. By the previous result, $\mathrm{id}_M^{*,k}\colon H^k M \to H^k(M)$ is the zero mapping for all $k > 0$. However, $\mathrm{id}_M^{*,k} = \mathrm{id}_{H^k(M)}$ for all k. $\square$

Poincaré lemma

A set $X \subseteq \mathbb{R}^n$ is *star-shaped* if there exists $c \in X$ such that for all $p \in X$, the line segment connecting c and p lies entirely in X.

Examples?

Proposition. If $U \subseteq \mathbb{R}^n$ is open and *star-shaped*, then $H^k U = 0$ for all $k > 0$.

Proof. Letting $c \in U$ be as in the above definition of *star-shaped*, the following homotopy shows that U is contractible:

$$\begin{aligned} h\colon [0,1] \times U &\to U \\ (t,x) &\mapsto (1-t)x + tc. \end{aligned}$$

□