Toric Varieties: Introduction

Lattice

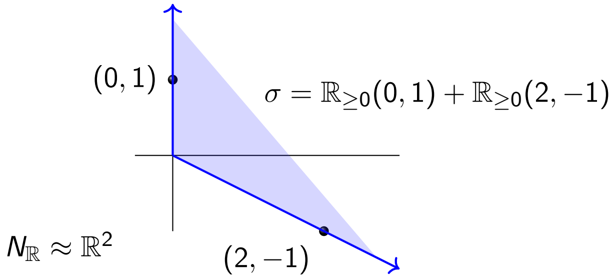
$N =$ free $\mathbb{Z}$-module of rank n

$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

A choice of a basis for N gives isomorphisms

$$N \approx \mathbb{Z}^n \quad \text{and} \quad N_{\mathbb{R}} \approx \mathbb{R}^n.$$

Cones



$\sigma \subset N_{\mathbb{R}} \approx \mathbb{R}^n$ = strongly convex, rational, polyhedral cone

cone: closed under addition and under multiplication by $\lambda \in \mathbb{R}_{\geq 0}$

strongly convex: contains no line through the origin

rational polyhedral: generated by a finite number of lattice points (elements of N)

Dual lattice

dual lattice: $M := \mathrm{Hom}_{\mathbb{Z}}(M, N)$

pairing:

$$\begin{array}{ccc} M \times N \to \mathbb{Z} & \overset{\otimes_{\mathbb{Z}}\mathbb{R}}{\rightsquigarrow} & M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R} \\ (\phi, v) \to \phi(v) & & (\phi, v) \to \phi(v) \end{array}$$

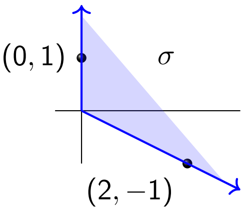
Choosing a basis $e_1, \dots, e_n$ for N and the dual basis $e_1^*, \dots, e_n^*$ for M:

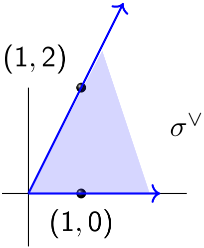
$$\begin{array}{ccc} \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z} & \overset{\otimes_{\mathbb{Z}}\mathbb{R}}{\rightsquigarrow} & \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \\ (u, v) \to u \cdot v & & (u, v) \to u \cdot v \end{array}$$

Dual cone

The *dual* of the cone $\sigma \subset N_{\mathbb{R}}$ is

$$\sigma^{\vee} := \{u \in M_{\mathbb{R}} : u(v) \geq 0 \text{ for all } v \in \sigma\} \subset M_{\mathbb{R}}.$$





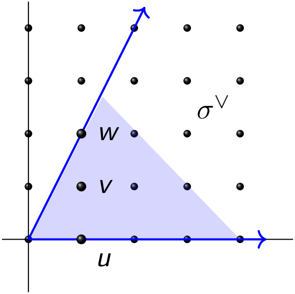
A *face* τ of σ is a set of the form

$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma : \langle u, v \rangle = 0\}$$

for some $u \in \sigma^{\vee}$. (τ is a cone; *facet* = face of codimension 1)

Semigroup: generators and relations

The *semigroup corresponding to* σ is $S_\sigma = \sigma^\vee \cap M$, the set of lattice points in $\sigma^\vee$.

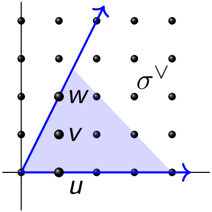


generators: $u = (1, 0)$, $v = (1, 1)$, $w = (1, 2)$

relations: $u + w = 2v$

Semigroup algebras and toric varieties

$\mathbb{C}[S_\sigma] := \mathbb{C}[e_u : u \in S_\sigma]$ where $e_u \cdot e_v := e_{u+v}$ and $e_0 = 1$.



$$w = (1,2) = 12$$
$$v = (1,1) = 11$$
$$u = (1,0) = 10$$
$$u + w = 2v$$

$$\mathbb{C}[x,y,z] \to \mathbb{C}[S_\sigma]$$
$$x \mapsto e_{10}$$
$$y \mapsto e_{11}$$
$$z \mapsto e_{12}$$

Yields: $\mathbb{C}[x,y,z]/(xz - y^2) \xrightarrow{\sim} \mathbb{C}[S_\sigma]$. The *toric variety* corresponding to σ is the solution set to $xz = y^2$ in $\mathbb{C}^3 \approx \mathbb{R}^6$.

Constructing an affine toric variety

$$\begin{aligned}
\sigma &\to \sigma^\vee \\
&\to S_\sigma = \sigma^\vee \cap M \\
&\to \mathbb{C}[S_\sigma] = \mathbb{C}[e_{u_1}, \ldots, e_{u_n}] \\
&\approx \mathbb{C}[x_1, \ldots, x_n]/I
\end{aligned}$$

$$I = (f_1, \ldots, f_k)$$

Toric variety corresponding to σ:

$$U_\sigma = \{x \in \mathbb{C}^n : f_1(x) = \cdots = f_k(x) = 0\}.$$

Ideals in polynomial rings

$I \subseteq \mathbb{C}[x_1, \ldots, x_n] =: R$ is an *ideal* if

1. $f, g \in I \Rightarrow f + g \in I$
2. $f \in I,\ g \in R \Rightarrow fg \in I$

Hilbert's basis theorem. There exists $f_1, \ldots, f_k \in I$ such that

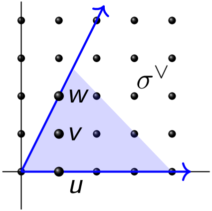
$$I = (f_1, \ldots, f_k) := \{\textstyle\sum_{i=1}^{k} g_i f_i : g_1, \ldots, g_k \in R\}.$$

Zero set of I:

$$\begin{aligned} Z(I) &= \{p \in \mathbb{C}^n : f(p) = 0 \text{ for all } f \in I\} \\ &= \{p \in \mathbb{C}^n : f_1(p) = \cdots = f_k(p) = 0\}. \end{aligned}$$

Review

$\mathbb{C}[S_\sigma] := \mathbb{C}[e_u : u \in S_\sigma]$ where $e_u \cdot e_v := e_{u+v}$ and $e_0 = 1$.



$$w = (1,2) = 12$$
$$v = (1,1) = 11$$
$$u = (1,0) = 10$$
$$u + w = 2v$$

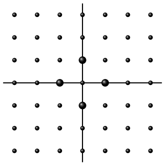
$\mathbb{C}[x,y,z] \to \mathbb{C}[S_\sigma]$ with $x \mapsto e_{10}$, $y \mapsto e_{11}$, $z \mapsto e_{12}$.

$\mathbb{C}[x,y,z]/(xz - y^2) \xrightarrow{\sim} \mathbb{C}[S_\sigma]$,

$U_\sigma = \{(x,y,z) \in \mathbb{C}^3 : xz - y^2 = 0\}$, a cone in 3-space

Example 2

$$\sigma = \{0\} \subset \mathbb{R}^2, \qquad \sigma^\vee = \mathbb{R}^2, \qquad S_\sigma = \sigma^\vee \cap M$$



generators: $u_1 = (1,0), v_1 = (-1,0), u_2 = (0,1), v_2 = (0,-1)$

relations: $u_1 + v_1 = 0,\ u_2 + v_2 = 0$

$$\mathbb{C}[x_1, x_2, y_1, y_2]/I \xrightarrow{\sim} \mathbb{C}[e_{u_1}, e_{u_2}, e_{v_1}, e_{v_2}]$$

$$I = (x_1 y_1 - 1, x_2 y_2 - 1)$$

$$U_\sigma = Z(I) = \{(x_1, y_1, x_2, y_2) \in \mathbb{C}^4 : x_1 y_1 = 1, x_2 y_2 = 1\}$$

Example 2

$$U_\sigma = \{(x_1, y_1, x_2, y_2) \in \mathbb{C}^4 : x_1 y_1 = 1, x_2 y_2 = 1\}$$
$$\approx \{(x_1, x_2) \in \mathbb{C}^2 : x_1 \neq 0, x_2 \neq 0\} = \mathbb{C}^* \times \mathbb{C}^* = (\mathbb{C}^*)^2$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Mapping:

$$U_\sigma \to (\mathbb{C}^*)^2$$
$$(x_1, y_1, x_2, y_2) \mapsto (x_1, x_2)$$

Inverse: $(x_1, x_2) \mapsto (x_1, 1/x_1, x_2, 1/x_2)$.

Terminology: $(\mathbb{C}^*)^2$ is the *algebraic* 2-*torus*

Example 3

$\sigma = \{0\} \subset \mathbb{R}^n, \qquad \sigma^\vee = \mathbb{R}^n,$

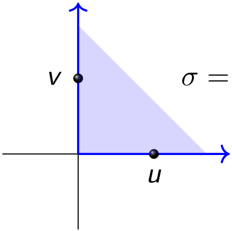
$S_\sigma = \sigma \cap M$:

generators: $u_i = e_i, v_i = -e_i$ for $i = 1, \ldots, n$.

relations: $u_i + v_i = 0$ for $i = 1, \ldots, n$.

$$\begin{aligned} U_\sigma &= Z(I) = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{C}^{2n} : x_i y_i = 1 \text{ for all } i\} \\ &\approx \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i \neq 0 \text{ for all } i\} \\ &= (\mathbb{C}^*)^n. \end{aligned}$$

Example 4



$\sigma = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1)$

$\sigma^{\vee} = \mathbb{R}_{\geq 0}(1,0) + \mathbb{R}_{\geq 0}(0,1)$

$S_\sigma = \sigma \cap M$:

generators: e_1, e_2 relations: none,

$\mathbb{C}[x,y] \xrightarrow{\sim} \mathbb{C}[e_{10}, e_{01}]$, $I = (0)$

$U_\sigma = Z(I) = \mathbb{C}^2$

Example 5

$\sigma = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} e_i, \qquad \sigma^\vee = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} e_i,$

$S_\sigma = \sigma \cap M$:

generators: $e_1, \ldots, e_n$, relations: none,

$\mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\sim} \mathbb{C}[e_{e_i} : i = 1, \ldots, n], \qquad I = (0)$

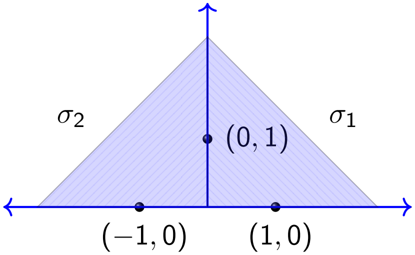
$U_\sigma = Z(I) = \mathbb{C}^n$

Fans

A fan, Δ is a set of s.c.r.p. cones such that:

1. $\sigma \in \Delta$ and τ a face of σ implies $\tau \in \Delta$
2. $\sigma, \tau \in \Delta$ implies $\sigma \cap \tau \in \Delta$.

Example 6



$\Delta:\ \sigma_1, \sigma_2$, and their faces (three rays and the origin)

$U_{\sigma_1} = \mathbb{C}^2$, coordinates: $x \mapsto e_{(1,0)}$, $y \mapsto e_{(0,1)}$

$U_{\sigma_2} = \mathbb{C}^2$, coordinates: $u \mapsto e_{(-1,0)}$, $v \mapsto e_{(0,1)}$

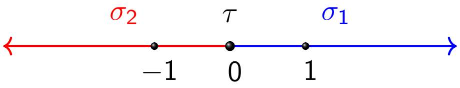
glue: $U_{\sigma_1} = \mathbb{C}^2_{x,y} \dashrightarrow \mathbb{C}^2_{u,v} = U_{\sigma_2}$, $x \mapsto \frac{1}{x}$, $y \mapsto y$

geometry: restricting to real points, we get $X(\Delta) \approx S^1 \times \mathbb{R}$, a cylinder

Example 7

$n = 1$: In dimension 1, there are only three possible cones:

$\sigma_1 = \mathbb{R}_{\geq 0}(1) = \mathbb{R}_{\geq 0}$, $\sigma_2 = \mathbb{R}_{\geq 0}(-1) = -\mathbb{R}_{\geq 0}$, and $\tau = \{0\}$.



$S_{\sigma_1} = \mathbb{Z}_{\geq 0} \cdot 1 = \mathbb{N}, \quad \mathbb{C}[x] \xrightarrow{\sim} \mathbb{C}[e_1] \quad U_{\sigma_1} = Z(I) = Z(0) = \mathbb{C}$

$S_{\sigma_2} = \mathbb{Z}_{\geq 0} \cdot (-1) = -\mathbb{N}, \quad \mathbb{C}[y] \xrightarrow{\sim} \mathbb{C}[e_{-1}] \quad U_{\sigma_2} = Z(0) = \mathbb{C}$

$S_\tau = \mathbb{Z}, \quad \mathbb{C}[x, y] \xrightarrow{\sim} \mathbb{C}[e_1, e_{-1}]$ with $x \mapsto e_1, y \mapsto e_{-1}$

$U_\tau = Z(xy - 1) = \{(x, y) \in \mathbb{C}^2 : xy = 1\} \xrightarrow{\sim} \{x \in \mathbb{C} : x \neq 0\}$.

Example 7

$$
\begin{array}{ccc}
\sigma_1 = \mathbb{N} & & \sigma_2 = -\mathbb{N} \\
& \nwarrow \quad \nearrow & \\
& \tau = \{0\} &
\end{array}
$$

$$
\begin{array}{ccc}
U_{\sigma_1} = \mathbb{C}_x & \overset{x \mapsto \frac{1}{x}}{\dashrightarrow} & U_{\sigma_2} = \mathbb{C}_y \\
& \nwarrow \qquad \nearrow & \\
& U_\tau = \{x \in \mathbb{C} : x \neq 0\} \xrightarrow{\sim} \{(x, y) \in \mathbb{C}^2 : xy = 1\} &
\end{array}
$$

Mappings of fans

Let Δ be a fan for the lattice N, and let Δ' be a fan for the lattice N'.

A mapping of fans $\Delta \to \Delta'$ is a $\mathbb{Z}$-linear mapping $\phi \colon N \to N'$ with the property that for each cone $\sigma \in \Delta$, there is a cone $\sigma' \in \Delta'$ such that $\phi(\sigma) \subseteq \sigma'$.

In that case, there is an induced mapping of toric varieties $X(\Delta) \to X(\Delta')$.

Example: Mappings of fans

$N = \mathbb{Z}$ and $N' = \mathbb{Z}^2$, $\quad \phi\colon \mathbb{Z} \to \mathbb{Z}^2$ with $\phi(a) = (a, 0)$.

$\Delta = \{\tau_1, \tau_2\}$ with $\tau_1 = \mathbb{R}_{\geq 0}(1)$, $\tau_2 = \mathbb{R}_{\geq 0}(-1)$.

$\Delta' = \{\sigma_1, \sigma_2\}$ with $\sigma_1 = \mathbb{R}^2_{\geq 0}(1,0) + \mathbb{R}^2_{\geq 0}(0,1)$ and $\sigma_2 = \mathbb{R}^2_{\geq 0}(-1,0) + \mathbb{R}^2_{\geq 0}(0,1)$. (See Example 6.)

$$(x, y) \longmapsto (1/x, y)$$

$$\begin{array}{cccccc}
(x,0) & U_{\sigma_1} & \dashrightarrow & U_{\sigma_2} & (z,0) \\
\Big\uparrow & \Big\uparrow & & \Big\uparrow & \Big\uparrow \\
x & U_{\tau_1} & \dashrightarrow & U_{\tau_2} & z \\
 & x & \longmapsto & \frac{1}{x} &
\end{array}$$

Example: projective plane $\mathbb{P}^2$

See our notes, Example 13.6. (The numbering might change with further updates to the notes.)