Homotopy invariance of de Rham cohomology

Wednesday quiz

1. State Stokes' theorem (with hypotheses).
2. Suppose that $M = \mathbb{R}^n_-$ and

$$\omega = \sum_{i=1}^{n} a_i \, dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n.$$

   Prove that $\int_M d\omega = \int_{\partial M} a_1(0, x_2, \ldots, x_n)$.

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$$f^* = g^* \colon H^k N \to H^k M$$

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Homotopy equivalence theorem. Let $f\colon M \to N$ and $g\colon N \to M$. Suppose that $g \circ f \sim \mathrm{id}_M \quad \text{and} \quad f \circ g \sim \mathrm{id}_N$. Then

$$H^k M \approx H^k N$$

for all k.

Example

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and define

$$\begin{aligned} \iota\colon S^1 &\to M \\ (x,y) &\mapsto (x,y) \end{aligned} \qquad\qquad \begin{aligned} r\colon M &\to S^1 \\ p &\mapsto \frac{p}{|p|}. \end{aligned}$$

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By homotopy equivalence, it follows that

$$H^k M \approx H^k S^1.$$

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Hence, $f^* = g^*$. $\square$

Prism operator

The key ingredient for constructing s is the *prism operator*:

$$\begin{aligned} P\colon \Omega^k([0,1]\times M) &\to \Omega^{k-1}M \\ \omega &\mapsto \int_{t=0}^{1} \omega\left(\frac{\partial}{\partial t}, -\right). \end{aligned}$$

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For each $t \in [0,1]$, define

$$\begin{aligned} i_t \colon M &\to [0,1] \times M \\ x &\mapsto (t, x). \end{aligned}$$

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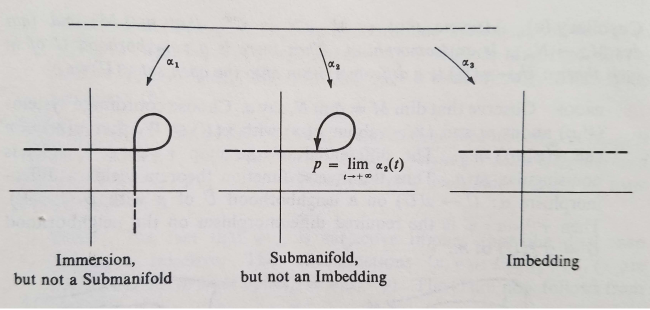
$= d(Ph^*) + (Ph^*)d = ds + sd$. (Note: d commutes with pullbacks.) $\square$

Immersions, submanifolds, embeddings



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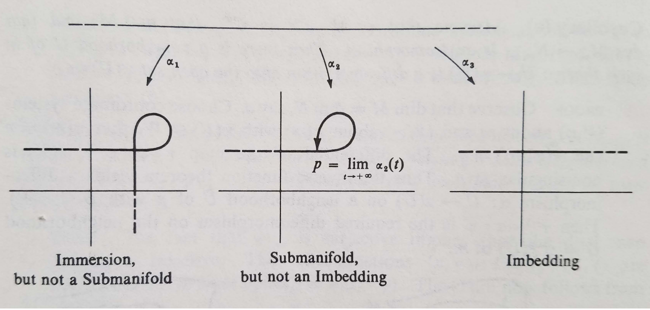

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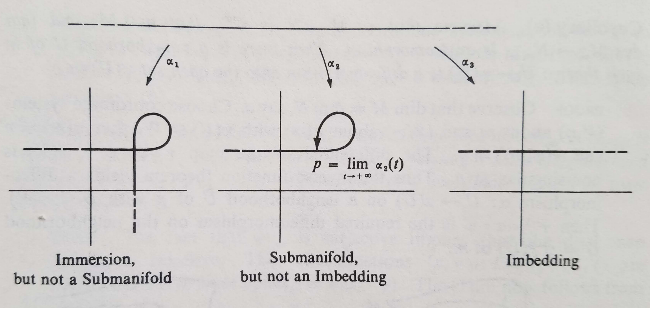

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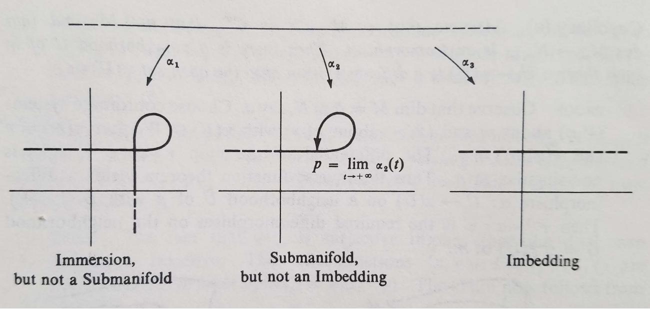

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- Immersion: $df_p : T_pM \to T_{f(p)}N$ injective for all p.
- Submanifold: f is an injective immersion.
- Embedding: f is an injective immersion and a homeomorphism onto $\mathrm{im}(f)$ with the subspace topology.

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Exercise. A deformation retraction is a homotopy equivalence, and thus, $H^k M \approx H^k S$ for all k.