Homotopy invariance of de Rham cohomology

Wednesday quiz

Don't forget the Wednesday quiz.

k-th de Rham cohomology group

$$\cdots \to \Omega^{k-1} M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1} M \to \cdots$$

 $H^k M := \ker d_k / \operatorname{im} d_{k-1}$

We've seen that

$$\mathcal{H}^k\mathbb{R} = egin{cases} \mathbb{R} & ext{if } k = 0 \ 0 & ext{if } k > 0, \end{cases}$$

and $H^n M \neq 0$ if M is a closed *n*-manifold.

Homotopy

Let $f, g: M \to N$. Then f is *homotopic* to g, denoted $f \sim g$ if there exists $h: [0, 1] \times M \to N$ such that

$$h(0, x) = f(x)$$

 $h(1, x) = g(x).$

Example. $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and

$$egin{aligned} h\colon [0,1] imes M & o M \ p\mapsto (1-t)p+t\,rac{p}{|p|} \end{aligned}$$

We have $h(0, x) = id_M(x)$.

Let r(x) = h(1, x). Then (i) $r: M \to S^1 \subset \mathbb{R}^2$, (ii) if $x \in S^1$, then r(x) = x, and (iii) $r \sim id_M$.

The mapping r is an example of a *deformation retraction*. We will see that this means $H^k M = H^k S^1$ for all k.

Homotopy invariance theorem

If $f,g\colon M o N$, and $f\sim g$, then $f^*=g^*\colon H^kN o H^kM$

for all k.

Homotopy equivalence

Let $f: M \to N$ and $g: N \to M$. Suppose that

 $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$.

Then, by the homotopy invariance theorem,

$$(g \circ f)^* = \mathrm{id}_M^* = \mathrm{id}_{H^k M} \colon H^k M \to H^k M$$

 $(f \circ g)^* = \mathrm{id}_N^* = \mathrm{id}_{H^k N} \colon H^k N \to H^k N.$

Thus,

$$(g \circ f)^* = f^* \circ g^* = \mathrm{id}_{H^k M}$$

 $(f \circ g)^* = g^* \circ f^* = \mathrm{id}_{H^k N}$

So f^* and g^* are inverse isomorphisms. Therefore, ...

Homotopy equivalence

Let $f: M \to N$ and $g: N \to M$. Suppose that

 $g \circ f \sim \operatorname{id}_M$ and $f \circ g \sim \operatorname{id}_N$.

Then

$$H^k M \approx H^k N$$

for all k.

Example

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and define $\iota \colon S^1 \to M$ $(x,y) \mapsto (x,y)$ $r \colon M \to S^1$ $p \mapsto \frac{p}{|p|}$.

Then, $r \circ \iota = \operatorname{id}_{S^1}$, and $\iota \circ r \stackrel{h}{\sim} \operatorname{id}_M$ by the *h* in our first example.

By homotopy equivalence, it follows that

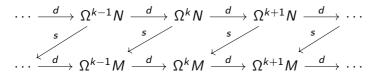
 $H^k M \approx H^k S^1.$

We will soon see (*Mayer-Vietoris*) that

$$H^k S^1 = \begin{cases} \mathbb{R} & \text{if } k = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof of homotopy invariance theorem

Idea: Find collection of mappings $s: \Omega^k N \to \Omega^{k-1} M$



such that for all $\omega \in \Omega^k N$,

$$g^*\omega - f^*\omega = (sd + ds)\omega.$$

If ω is closed, i.e., $d\omega = 0$, then $g^*\omega - f^*\omega = (sd + ds)\omega = d(s\omega)$. Therefore, $f^*\omega$ and g^* differ by an element in the image of d_{k-1} . So $[f^*\omega] = [g^*\omega]$ for all ω .

Hence, $f^* = g^*$.

Prism operator

The key ingredient for constructing *s* is the *prism operator*.

$$P: \Omega^{k}([0,1] \times M) \to \Omega^{k-1}M$$
$$\omega \mapsto \int_{t=0}^{1} \omega\left(\frac{\partial}{\partial t},-\right).$$

Example. $\omega = (x + 3t^2) dt \wedge dx \in \Omega^2([0, 1] \times \mathbb{R}).$

$$P\omega = \left(\int_{t=0}^{1} (x+3t^2) \, dt\right) \wedge dx = (x+1) \, dx.$$

(In general, integrate with respect to t and then drop the dt.)

If
$$\omega$$
 does not involve dt , then $P\omega = 0$, e.g. if
 $w = (x + y) dx \wedge dy \in \Omega^2([0, 1] \times \mathbb{R}^2)$, then $P\omega = 0$.

Proof of homotopy invariance theorem

For each $t \in [0, 1]$, define

$$egin{aligned} & i_t \colon M o [0,1] imes M \ & x \mapsto (t,x). \end{aligned}$$

We have that $(h \circ i_0) = f$ and $(h \circ \iota_1) = g$.

A straightforward calculation in coordinates (see our text) gives

$$\iota_1^*-\iota_0^*=dP-Pd.$$

Define $s = P \circ h^*$:

$$\Omega^k N \xrightarrow{h^*} \Omega^k([0,1] \times M) \xrightarrow{P} \Omega^{k-1} M,$$

Then $g^* - f^* = i_1^* h^* - i_0^* h^* = (i_1^* - i_0^*) h^* = (dP + Pd) h^*$ = $d(Ph^*) + (Ph^*)d = ds + sd$. (Note: *d* commutes with pullbacks.)