Homotopy invariance of de Rham cohomology

Wednesday quiz

Don't forget the Wednesday quiz.

k-th de Rham cohomology group

$$\cdots \to \Omega^{k-1}M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1}M \to \cdots$$

$$H^k M := \ker d_k / \operatorname{im} d_{k-1}$$

We've seen that

$$H^k\mathbb{R} = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k > 0, \end{cases}$$

and $H^n M \neq 0$ if M is a closed n-manifold.

Homotopy

Let $f, g \colon M \to N$. Then f is *homotopic* to g, denoted $f \sim g$ if there exists $h \colon [0,1] \times M \to N$ such that

$$h(0,x) = f(x)$$
$$h(1,x) = g(x).$$

Example. $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and

$$h \colon [0,1] \times M \to M$$
$$p \mapsto (1-t)p + t\,\frac{p}{|p|}.$$

We have $h(0,x) = \mathrm{id}_M(x)$.

Let $r(x) = h(1,x)$. Then (i) $r \colon M \to S^1 \subset \mathbb{R}^2$, (ii) if $x \in S^1$, then $r(x) = x$, and (iii) $r \sim \mathrm{id}_M$.

The mapping r is an example of a *deformation retraction*. We will see that this means $H^k M = H^k S^1$ for all k.

Homotopy invariance theorem

If $f, g\colon M \to N$, and $f \sim g$, then

$$f^* = g^*\colon H^k N \to H^k M$$

for all k.

Homotopy equivalence

Let $f\colon M \to N$ and $g\colon N \to M$. Suppose that

$$g \circ f \sim \mathrm{id}_M \quad \text{and} \quad f \circ g \sim \mathrm{id}_N\,.$$

Then, by the homotopy invariance theorem,

$$(g \circ f)^* = \mathrm{id}_M^* = \mathrm{id}_{H^k M}\colon H^k M \to H^k M$$

$$(f \circ g)^* = \mathrm{id}_N^* = \mathrm{id}_{H^k N}\colon H^k N \to H^k N.$$

Thus,

$$(g \circ f)^* = f^* \circ g^* = \mathrm{id}_{H^k M}$$

$$(f \circ g)^* = g^* \circ f^* = \mathrm{id}_{H^k N}$$

So f^* and g^* are inverse isomorphisms. Therefore, . . .

Homotopy equivalence

Let $f\colon M \to N$ and $g\colon N \to M$. Suppose that

$$g \circ f \sim \mathrm{id}_M \quad \text{and} \quad f \circ g \sim \mathrm{id}_N\,.$$

Then

$$H^k M \approx H^k N$$

for all k.

Example

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$, and define

$$\begin{aligned} \iota\colon S^1 &\to M \\ (x,y) &\mapsto (x,y) \end{aligned} \qquad\qquad \begin{aligned} r\colon M &\to S^1 \\ p &\mapsto \frac{p}{|p|}. \end{aligned}$$

Then, $r \circ \iota = \mathrm{id}_{S^1}$, and $\iota \circ r \overset{h}{\sim} \mathrm{id}_M$ by the h in our first example.

By homotopy equivalence, it follows that

$$H^k M \approx H^k S^1.$$

We will soon see ($\star$Mayer-Vietoris$\star$) that

$$H^k S^1 = \begin{cases} \mathbb{R} & \text{if } k = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof of homotopy invariance theorem

Idea: Find collection of mappings $s\colon \Omega^k N \to \Omega^{k-1} M$

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}N & \xrightarrow{d} & \Omega^{k}N & \xrightarrow{d} & \Omega^{k+1}N & \xrightarrow{d} & \cdots \\ & \swarrow^{s} & & \swarrow^{s} & & \swarrow^{s} & & \swarrow^{s} & \\ \cdots & \xrightarrow{d} & \Omega^{k-1}M & \xrightarrow{d} & \Omega^{k}M & \xrightarrow{d} & \Omega^{k+1}M & \xrightarrow{d} & \cdots \end{array}$$

such that for all $\omega \in \Omega^k N$,

$$g^*\omega - f^*\omega = (sd + ds)\omega.$$

If ω is closed, i.e., $d\omega = 0$, then $g^*\omega - f^*\omega = (sd + ds)\omega = d(s\omega)$. Therefore, $f^*\omega$ and g^* differ by an element in the image of d_{k-1}. So $[f^*\omega] = [g^*\omega]$ for all ω.

Hence, $f^* = g^*$. $\square$

Prism operator

The key ingredient for constructing s is the *prism operator*:

$$\begin{aligned} P\colon \Omega^k([0,1]\times M) &\to \Omega^{k-1}M \\ \omega &\mapsto \int_{t=0}^{1} \omega\left(\frac{\partial}{\partial t}, -\right). \end{aligned}$$

Example. $\omega = (x+3t^2)\, dt \wedge dx \in \Omega^2([0,1]\times\mathbb{R})$.

$$P\omega = \left(\int_{t=0}^{1} (x+3t^2)\, dt\right) \wedge dx = (x+1)\, dx.$$

(In general, integrate with respect to t and then drop the dt.)

If ω does not involve dt, then $P\omega = 0$, e.g. if $w = (x+y)\, dx \wedge dy \in \Omega^2([0,1]\times\mathbb{R}^2)$, then $P\omega = 0$.

Proof of homotopy invariance theorem

For each $t \in [0, 1]$, define

$$\begin{aligned} i_t \colon M &\to [0,1] \times M \\ x &\mapsto (t, x). \end{aligned}$$

We have that $(h \circ i_0) = f$ and $(h \circ \iota_1) = g$.

A straightforward calculation in coordinates (see our text) gives

$$\iota_1^* - \iota_0^* = dP - Pd.$$

Define $s = P \circ h^*$:

$$\Omega^k N \xrightarrow{h^*} \Omega^k([0,1] \times M) \xrightarrow{P} \Omega^{k-1} M,$$

Then $g^* - f^* = i_1^* h^* - i_0^* h^* = (i_1^* - i_0^*) h^* = (dP + Pd) h^*$
$= d(Ph^*) + (Ph^*)d = ds + sd$. (Note: d commutes with pullbacks.) □