Exact sequences

Exact sequence

A sequence of linear mappings $U \xrightarrow{f} V \xrightarrow{g} W$ of vector spaces is *exact* at V if $\operatorname{im}(f) = \ker(g)$.

A sequence

$$\cdots \xrightarrow{d_{k-2}} V_{k-1} \xrightarrow{d_{k-1}} V_k \xrightarrow{d_k} V_{k+1} \xrightarrow{d_{k+1}}$$

is *exact* if it is exact at each V_k.

A *short exact sequence* is an exact sequence of the form

$$0 \to U \to V \to W \to 0.$$

Short exact sequence

Suppose we have a short exact sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0.$$

Then

- f is injective and g is surjective.
- There is an isomorphism $U \to \ker(g)$ defined by $u \mapsto f(u)$. (Why is this an isomorphism?)
- By definition, the *cokernel* of f is $\operatorname{cok}(f) = V / \operatorname{im}(f)$. There is an isomorphism $\operatorname{cok}(f) \to W$ given by $[v] \mapsto g(v)$.
- We have $\dim U + \dim W = \dim V$.
- For an exact sequence $0 \to V_1 \to V_2 \to \cdots \to V_k \to 0$, we have $\sum_{i=1}^{k}(-1)^i \dim V_i = 0$.

Snake lemma

Suppose the following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C &
\end{tikzcd}$$

Snake lemma

$$
\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] & \\
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{cok} f \arrow[r] & \operatorname{cok} g \arrow[r] & \operatorname{cok} h &
\end{tikzcd}
$$

Snake lemma

$$
\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners, "\partial"] & \\
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{cok} f \arrow[r] & \operatorname{cok} g \arrow[r] & \operatorname{cok} h &
\end{tikzcd}
$$

Snake lemma

Given a commutatitive diagram with exact rows:

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C &
\end{tikzcd}$$

There exists a *connecting homomorphism* $\partial\colon \ker h \to \operatorname{cok} f$ so that the following sequence is exact:

$$\ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{cok}\ f \to \operatorname{cok}\ g \to \operatorname{cok}\ h.$$

Snake lemma

Given a commutatitive diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

There exists a *connecting homomorphism* $\partial\colon \ker h \to \operatorname{cok} f$ so that the following sequence is exact:

$$0 \to \ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{cok} f \to \operatorname{cok} g \to \operatorname{cok} h \to 0.$$

Category of cochain complexes

Cochain complex:

$$A^\bullet : \quad \cdots \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} \cdots \qquad (d^2 = 0)$$

Mappings $f \colon A^\bullet \to B^\bullet$:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & A^k \arrow[r, "d"] \arrow[d, "f_k"'] & A^{k+1} \arrow[r, "d"] \arrow[d, "f_{k+1}"'] & \cdots \\
\cdots \arrow[r, "d"] & B^k \arrow[r, "d"] & B^{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

where the mappings commute.

Cohomology of cochain complexes

Cochain complex:

$$A^\bullet : \quad \cdots \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} \cdots \qquad (d^2 = 0)$$

Cohomology:

$$H^k A := \ker(d_k)/\operatorname{im}(d_{k-1}).$$

Short exact sequence of cochain complexes

A *short exact sequence of cochain complexes* is a sequence of mappings

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0.$$

such that

$$0 \to A^k \to B^k \to C^k \to 0$$

is exact for all k.

Theorem. A short exact sequence of cochain complexes induces a long exact sequence in cohomology:

$$\cdots \to H^{k-1}C \xrightarrow{\partial} H^k A \to H^k B \to H^k C \xrightarrow{\partial} H^{k+1}A \to \cdots$$

Long exact sequence in cohomology: proof of existence

Apply the snake lemma to:

$$\begin{tikzcd}
0 \arrow[r] & A^k \arrow[r] \arrow[d, "d_k^A"] & B^k \arrow[r] \arrow[d, "d_k^B"] & C^k \arrow[r] \arrow[d, "d_k^C"] & 0 \\
0 \arrow[r] & A^{k+1} \arrow[r] & B^{k+1} \arrow[r] & C^{k+1} \arrow[r] & 0
\end{tikzcd}$$

to get an exact sequence

$$0 \to \ker d_k^A \to \ker d_k^B \to \ker d_k^C \xrightarrow{\partial} \mathrm{cok}\ d_k^A \to \mathrm{cok}\ d_k^B \to \mathrm{cok}\ d_k^C \to 0.$$

Long exact sequence in cohomology: proof of existence

The mapping $d_k^A \colon A^k \to A^{k+1}$ induces

$$\begin{aligned} A^k / \operatorname{im} d_{k-1}^A &\to A^{k+1} \\ [a] &\mapsto d_k^A(a). \end{aligned}$$

and

$$\begin{aligned} A^k / \operatorname{im} d_{k-1}^A &\to \ker d_{k+1}^A \\ [a] &\mapsto d_k^A(a). \end{aligned}$$

Long exact sequence in cohomology: proof of existence

The mapping $d^A_k \colon A^k \to A^{k+1}$ induces

$$\begin{aligned} A^k / \operatorname{im} d^A_{k-1} &\to A^{k+1} \\ [a] &\mapsto d^A_k(a). \end{aligned}$$

and

$$\begin{aligned} \operatorname{cok} d^A_{k-1} := A^k / \operatorname{im} d^A_{k-1} &\to \ker d^A_{k+1} \\ [a] &\mapsto d^A_k(a). \end{aligned}$$

So define

$$\begin{aligned} \delta^A_k \colon \operatorname{cok} d^A_{k-1} &\to \ker d^A_{k+1} \\ [a] &\mapsto d^A_k(a). \end{aligned}$$

Long exact sequence in cohomology: proof of existence

We have

$$\begin{aligned}\delta_k^A\colon\ \operatorname{cok} d_{k-1}^A = A^k / \operatorname{im} d_{k-1}^A &\to \ker d_{k+1}^A \\ [a] &\mapsto d_k^A(a).\end{aligned}$$

Then

$$\operatorname{cok} \delta_k^A := \ker d_{k+1}^A / \operatorname{im} \delta_k^A = \ker d_{k+1}^A / \operatorname{im} d_k^A = H^{k+1}A.$$

and

$$\ker \delta_k^A = \ker d_k^A / \operatorname{im} d_{k-1}^A = H^k A.$$

Long exact sequence in cohomology: proof of existence

Our earlier application of the snake lemma gave us, for each k, and exact sequence,

$$0 \to \ker d_k^A \to \ker d_k^B \to \ker d_k^C \xrightarrow{\partial} \text{cok } d_k^A \to \text{cok } d_k^B \to \text{cok } d_k^C \to 0.$$

Therefore, the following commutative diagram has exact rows:

$$\begin{tikzcd}
 & \text{cok}\, d_{k-1}^A \arrow[r] \arrow[d, "\delta_k^A"] & \text{cok}\, d_{k-1}^B \arrow[r] \arrow[d, "\delta_k^B"] & \text{cok}\, d_{k-1}^C \arrow[r] \arrow[d, "\delta_k^C"] & 0 \\
0 \arrow[r] & \ker d_{k+1}^A \arrow[r] & \ker d_{k+1}^B \arrow[r] & \ker d_{k+1}^C &
\end{tikzcd}$$

Long exact sequence in cohomology: proof of existence

Apply the snake lemma to

$$\begin{tikzcd}
& \operatorname{cok} d^A_{k-1} \arrow[r] \arrow[d, "\delta^A_k"] & \operatorname{cok} d^B_{k-1} \arrow[r] \arrow[d, "\delta^B_k"] & \operatorname{cok} d^C_{k-1} \arrow[r] \arrow[d, "\delta^C_k"] & 0 \\
0 \arrow[r] & \ker d^A_{k+1} \arrow[r] & \ker d^B_{k+1} \arrow[r] & \ker d^C_{k+1} &
\end{tikzcd}$$

to get the exact sequence in cohomology:

$$H^k A \to H^k B \to H^k C \xrightarrow{\partial} H^{k+1} A \to H^{k+1} B \to H^{k+1} C$$

Piece these together over all k to get the long exact sequence in homology

$$\cdots \to H^{k-1} C \xrightarrow{\partial} H^k A \to H^k B \to H^k C \xrightarrow{\partial} H^{k+1} A \to \cdots$$

Long exact sequence in cohomology

Theorem. A short exact sequence of cochain complexes

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0.$$

induces a long exact sequence in cohomology:

$$\cdots \to H^{k-1}C \xrightarrow{\partial} H^k A \to H^k B \to H^k C \xrightarrow{\partial} H^{k+1}A \to \cdots$$