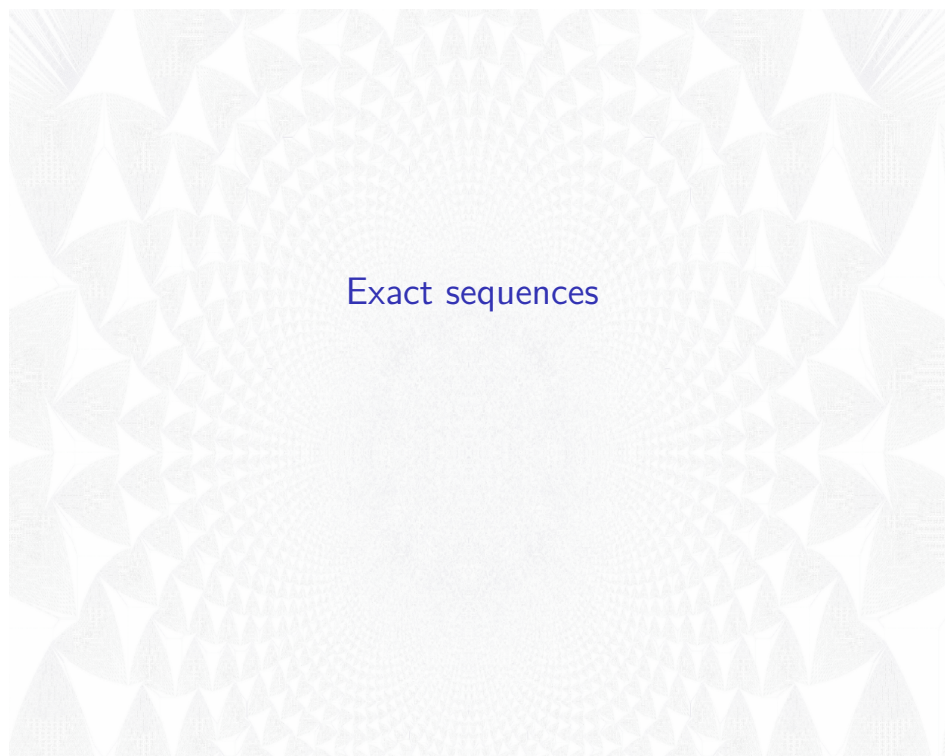




## Exact sequences

## Exact sequence

A sequence of linear mappings  $U \xrightarrow{f} V \xrightarrow{g} W$  of vector spaces is *exact* at  $V$  if  $\text{im}(f) = \ker(g)$ .

A sequence

$$\cdots \xrightarrow{d_{k-2}} V_{k-1} \xrightarrow{d_{k-1}} V_k \xrightarrow{d_k} V_{k+1} \xrightarrow{d_{k+1}}$$

is *exact* if it is exact at each  $V_k$ .

A *short exact sequence* is an exact sequence of the form

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0.$$

## Short exact sequence

Suppose we have a short exact sequence

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0.$$

Then

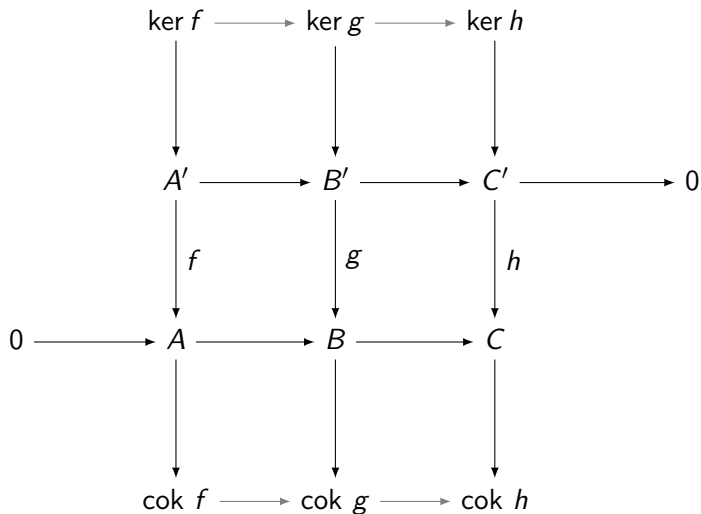
- ▶  $f$  is injective and  $g$  is surjective.
- ▶ There is an isomorphism  $U \rightarrow \ker(g)$  defined by  $u \mapsto f(u)$ . (Why is this an isomorphism?)
- ▶ By definition, the *cokernel* of  $f$  is  $\text{cok}(f) = V/\text{im}(f)$ . There is an isomorphism  $\text{cok}(f) \rightarrow W$  given by  $[v] \mapsto g(v)$ .
- ▶ We have  $\dim U + \dim W = \dim V$ .
- ▶ For an exact sequence  $0 \rightarrow V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_k \rightarrow 0$ , we have  $\sum_{i=1}^k (-1)^i \dim V_i = 0$ .

## Snake lemma

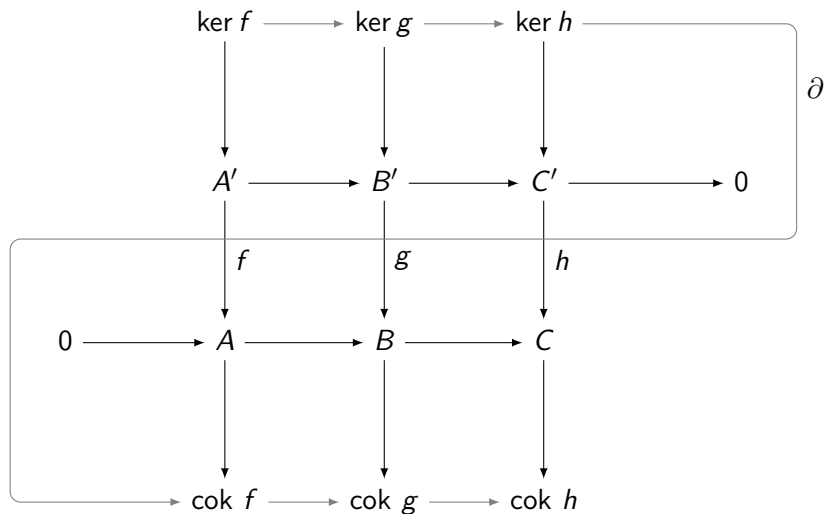
Suppose the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow 0 \\ & \downarrow f & & \downarrow g & & \downarrow h & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \end{array}$$

## Snake lemma



## Snake lemma



## Snake lemma

Given a commutative diagram with exact rows:

$$\begin{array}{ccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \end{array}$$

There exists a *connecting homomorphism*  $\partial: \ker h \rightarrow \operatorname{cok} f$  so that the following sequence is exact:

$$\ker f \rightarrow \ker g \rightarrow \ker h \xrightarrow{\partial} \operatorname{cok} f \rightarrow \operatorname{cok} g \rightarrow \operatorname{cok} h.$$

## Snake lemma

Given a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

There exists a *connecting homomorphism*  $\partial: \ker h \rightarrow \operatorname{cok} f$  so that the following sequence is exact:

$$0 \rightarrow \ker f \rightarrow \ker g \rightarrow \ker h \xrightarrow{\partial} \operatorname{cok} f \rightarrow \operatorname{cok} g \rightarrow \operatorname{cok} h \rightarrow 0.$$



# Category of cochain complexes

Cochain complex:

$$A^\bullet: \quad \dots \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} \dots \quad (d^2 = 0)$$

Mappings  $f: A^\bullet \rightarrow B^\bullet$ :

$$\begin{array}{ccccccc} \dots & \xrightarrow{d} & A^k & \xrightarrow{d} & A^{k+1} & \xrightarrow{d} & \dots \\ & & \downarrow f_k & & \downarrow f_{k+1} & & \\ \dots & \xrightarrow{d} & B^k & \xrightarrow{d} & B^{k+1} & \xrightarrow{d} & \dots \end{array}$$

where the mappings commute.

# Cohomology of cochain complexes

Cochain complex:

$$A^\bullet : \quad \dots \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} \dots \quad (d^2 = 0)$$

Cohomology:

$$H^k A := \ker(d_k) / \operatorname{im}(d_{k-1}).$$

## Short exact sequence of cochain complexes

A *short exact sequence of cochain complexes* is a sequence of mappings

$$0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0.$$

such that

$$0 \rightarrow A^k \rightarrow B^k \rightarrow C^k \rightarrow 0$$

is exact for all  $k$ .

**Theorem.** A short exact sequence of cochain complexes induces a long exact sequence in cohomology:

$$\dots \rightarrow H^{k-1}C \xrightarrow{\partial} H^kA \rightarrow H^kB \rightarrow H^kC \xrightarrow{\partial} H^{k+1}A \rightarrow \dots$$

# Long exact sequence in cohomology: proof of existence

Apply the snake lemma to:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^k & \longrightarrow & B^k & \longrightarrow & C^k \longrightarrow 0 \\ & & \downarrow d_k^A & & \downarrow d_k^B & & \downarrow d_k^C \\ 0 & \longrightarrow & A^{k+1} & \longrightarrow & B^{k+1} & \longrightarrow & C^{k+1} \longrightarrow 0 \end{array}$$

to get an exact sequence

$$0 \rightarrow \ker d_k^A \rightarrow \ker d_k^B \rightarrow \ker d_k^C \xrightarrow{\partial} \operatorname{cok} d_k^A \rightarrow \operatorname{cok} d_k^B \rightarrow \operatorname{cok} d_k^C \rightarrow 0.$$

# Long exact sequence in cohomology: proof of existence

The mapping  $d_k^A: A^k \rightarrow A^{k+1}$  induces

$$\begin{aligned} A^k / \operatorname{im} d_{k-1}^A &\rightarrow A^{k+1} \\ [a] &\mapsto d_k^A(a). \end{aligned}$$

and

$$\begin{aligned} A^k / \operatorname{im} d_{k-1}^A &\rightarrow \ker d_{k+1}^A \\ [a] &\mapsto d_k^A(a). \end{aligned}$$

# Long exact sequence in cohomology: proof of existence

The mapping  $d_k^A: A^k \rightarrow A^{k+1}$  induces

$$A^k / \operatorname{im} d_{k-1}^A \rightarrow A^{k+1} \\ [a] \mapsto d_k^A(a).$$

and

$$\operatorname{cok} d_{k-1}^A := A^k / \operatorname{im} d_{k-1}^A \rightarrow \ker d_{k+1}^A \\ [a] \mapsto d_k^A(a).$$

So define

$$\delta_k^A: \operatorname{cok} d_{k-1}^A \rightarrow \ker d_{k+1}^A \\ [a] \mapsto d_k^A(a).$$

## Long exact sequence in cohomology: proof of existence

We have

$$\begin{aligned}\delta_k^A: \operatorname{cok} d_{k-1}^A = A^k / \operatorname{im} d_{k-1}^A &\rightarrow \operatorname{ker} d_{k+1}^A \\ [a] &\mapsto d_k^A(a).\end{aligned}$$

Then

$$\operatorname{cok} \delta_k^A := \operatorname{ker} d_{k+1}^A / \operatorname{im} \delta_k^A = \operatorname{ker} d_{k+1}^A / \operatorname{im} d_k^A = H^{k+1} A.$$

and

$$\operatorname{ker} \delta_k^A = \operatorname{ker} d_k^A / \operatorname{im} d_{k-1}^A = H^k A.$$

## Long exact sequence in cohomology: proof of existence

Our earlier application of the snake lemma gave us, for each  $k$ , and exact sequence,

$$0 \rightarrow \ker d_k^A \rightarrow \ker d_k^B \rightarrow \ker d_k^C \xrightarrow{\partial} \operatorname{cok} d_k^A \rightarrow \operatorname{cok} d_k^B \rightarrow \operatorname{cok} d_k^C \rightarrow 0.$$

Therefore, the following commutative diagram has exact rows:

$$\begin{array}{ccccccc} \operatorname{cok} d_{k-1}^A & \longrightarrow & \operatorname{cok} d_{k-1}^B & \longrightarrow & \operatorname{cok} d_{k-1}^C & \longrightarrow & 0 \\ \downarrow \delta_k^A & & \downarrow \delta_k^B & & \downarrow \delta_k^C & & \\ 0 & \longrightarrow & \ker d_{k+1}^A & \longrightarrow & \ker d_{k+1}^B & \longrightarrow & \ker d_{k+1}^C \end{array}$$



## Long exact sequence in cohomology: proof of existence

Apply the snake lemma to

$$\begin{array}{ccccccc} \operatorname{cok} d_{k-1}^A & \longrightarrow & \operatorname{cok} d_{k-1}^B & \longrightarrow & \operatorname{cok} d_{k-1}^C & \longrightarrow & 0 \\ \downarrow \delta_k^A & & \downarrow \delta_k^B & & \downarrow \delta_k^C & & \\ 0 & \longrightarrow & \ker d_{k+1}^A & \longrightarrow & \ker d_{k+1}^B & \longrightarrow & \ker d_{k+1}^C \end{array}$$

to get the exact sequence in cohomology:

$$H^k A \rightarrow H^k B \rightarrow H^k C \xrightarrow{\partial} H^{k+1} A \rightarrow H^{k+1} B \rightarrow H^{k+1} C$$

Piece these together over all  $k$  to get the long exact sequence in homology

$$\dots \rightarrow H^{k-1} C \xrightarrow{\partial} H^k A \rightarrow H^k B \rightarrow H^k C \xrightarrow{\partial} H^{k+1} A \rightarrow \dots$$

## Long exact sequence in cohomology

**Theorem.** A short exact sequence of cochain complexes

$$0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0.$$

induces a long exact sequence in cohomology:

$$\dots \rightarrow H^{k-1}C \xrightarrow{\partial} H^kA \rightarrow H^kB \rightarrow H^kC \xrightarrow{\partial} H^{k+1}A \rightarrow \dots$$