Stokes' theorem

Wednesday quiz

Don't forget Wednesday's quiz.

Manifolds with boundary

Definition 10.16. An n-dimensional *manifold with boundary* is a second-countable, Hausdorff topological space M that is locally homeomorphic to open subsets of $\mathbb{R}^n_-$ with differentiable transition functions.

A point $p \in M$ is in the *boundary* of M if there is some (hence every) chart (U, h) at p such that $h(p) \in \partial h(U) \subseteq \mathbb{R}^n_-$. The collection of all such points is denoted ∂M.

Manifolds with boundary, tangent space

Let $p \in \partial M$, and let (U, h) be a chart at p. Define

$$T_p^- M = dh_p^{-1}(\mathbb{R}^n_-), \quad T_p^+ M = dh_p^{-1}(\mathbb{R}^n_+).$$

The inclusion mapping $\iota \colon \partial M \hookrightarrow M$ induces an inclusion $d\iota_p \colon T_p \partial M \hookrightarrow T_p M$. We have

$$T_p \partial M = T_p^- M \cap T_p^+ M.$$

inward-pointing tangent vectors: $T_p^- M \setminus T_p \partial M$

outward-pointing tangent vectors: $T_p^+ M \setminus T_p \partial M$.

Manifolds with boundary

Definition. Let M be an n-dimensional oriented manifold with boundary and let $p \in \partial M$. We define the *natural orientation* on ∂M as follows:

An ordered basis $\langle w_1, \ldots, w_{n-1} \rangle$ for $T_p \partial M$ is positively oriented if for any outward-pointing tangent vector $v \in T_p M$, the ordered basis $\langle v, w_1, \ldots, w_{n-1} \rangle$ for $T_p M$ is positively oriented in $T_p M$.

Stokes' theorem

Let M be an oriented n-manifold with boundary, and let $\omega \in \Omega^{n-1} M$ be a form with compact support. Then

$$\int_M d\omega = \int_{\partial M} \omega = \int_{\partial M} \iota^* \omega$$

where $\iota \colon \partial M \to M$ is the inclusion mapping.

Example

$$\int_M d\omega = \int_{\partial M} \omega$$

Example. Let $f\colon [0,1] \to \mathbb{R}$. Think of f as a zero-form on the manifold $M = [0,1]$, i.e., $f \in \Omega^0[0,1]$. Then

$$\int_M df = \int_{[0,1]} f'(x)\, dx = \int_{x=0}^{1} f'(x) = f(1) - f(0) = \int_{\partial[0,1]} f.$$

For the last equality, we need a definition of the oriented boundary of a one-dimensional manifold.

Examples

$n = 1$: The flow of a gradient vector field along a curve C is given by the change in the potential, ϕ:

$$\int_C \nabla\phi = \phi(C(1)) - \phi(C(0)).$$

$n = 2$: The flux of the curl of a vector field F through a surface S equals the flow of the vector field along the boundary of S:

$$\int_S \mathrm{curl}(F) \cdot \vec{n} = \int_{\partial S} F \cdot \vec{t}$$

$n = 3$: The integral of the divergence of a vector field F over a solid V equals the flux of the vector field through the boundary of V:

$$\int_V \mathrm{div}(F) = \int_{\partial V} F \cdot \vec{n}.$$

Proof of Stokes' theorem

Case 1. $M = \mathbb{R}^n_-, \quad \omega = \sum_{i=1}^n a_i \, dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n.$

$$d\omega = \sum_{i=1}^n \left(\frac{\partial a_i}{\partial x_1} dx_1 + \cdots + \frac{\partial a_i}{\partial x_n} dx_n \right) \wedge \, dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n \frac{\partial a_i}{\partial x_i} dx_i \wedge dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^n (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n.$$

Therefore,

$$\int_M d\omega = \sum_{i=1}^n (-1)^{i-1} \int_{\mathbb{R}^n_-} \frac{\partial a_i}{\partial x_i}.$$

Apply Fubini's theorem

For $i > 1$:

$$\begin{aligned}\int_{x_i=-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} &= \lim_{t\to\infty}\int_0^t \frac{\partial a_i}{\partial x_i} + \lim_{t\to-\infty}\int_t^0 \frac{\partial a_i}{\partial x_i} \\ &= \lim_{t\to\infty}(a_i(\cdots, t, \cdots)) - a_i(\cdots, 0, \cdots)) \\ &\quad + \lim_{t\to-\infty}(a_i(\cdots, 0, \cdots)) - a_i(\cdots, t, \cdots)) \\ &= 0\end{aligned}$$

since $\mathrm{supp}(\omega)$ is compact.

Thus,

$$\int_{\mathbb{R}^n_-} \frac{\partial a_i}{\partial x_i} = 0$$

for $i > 1$.

Apply Fubini's theorem

For $i = 1$:

$$\int_{x_1=-\infty}^{0} \frac{\partial a_1}{\partial x_1} = \lim_{t \to -\infty} (a_1(0, x_2, \ldots, x_n) - a_1(t, x_2, \ldots, x_n))$$

$$= a_1(0, x_2, \ldots, x_n).$$

Proof of Stokes' theorem, Case 1

$$\int_{\mathbb{R}^n_-} d\omega = \int \sum_{i=1}^n (-1)^{i-1} \frac{\partial a_i}{\partial x_i}$$

$$= \int_{\mathbb{R}^n_-} \frac{\partial a_1}{\partial x_1}$$

$$= \int_{\partial \mathbb{R}^n_-} \int_{x_1=-\infty}^0 \frac{\partial a_i}{\partial x_i}$$

$$= \int_{\partial \mathbb{R}^n_-} a_1(0, x_2, \ldots, x_n).$$

Thus, for the left-hand side of Stokes' theorem:

$$\int_M d\omega = \int_{\partial \mathbb{R}^n_-} a_1(0, x_2, \ldots, x_n).$$

Proof of Stokes' theorem, Case 1

For the right-hand side:

$$\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial \mathbb{R}^n_-} \iota^* \omega \\
&= \int_{\partial \mathbb{R}^n_-} \iota^* \textstyle\sum_{i=1}^n a_i dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n \\
&= \int_{\partial \mathbb{R}^n_-} \textstyle\sum_{i=1}^n (a_i \circ \iota) d\iota_1 \wedge \cdots \wedge \overline{d\iota_i} \wedge \cdots \wedge d\iota_n \\
&= \int_{\partial \mathbb{R}^n_-} a_1 \circ \iota \\
&= \int_{\partial \mathbb{R}^n_-} a_1(0, x_2, \ldots, x_n) \\
&= \int_{\mathbb{R}^n_-} d\omega = \int_M d\omega.
\end{aligned}$$

Proof of Stokes' theorem, Case 2

Suppose there exists a chart (U, h) such that $\operatorname{supp}(\omega) \subseteq U$.

Then use h to reduce to Case 1:

We may assume that $M = U \subseteq \mathbb{R}^n_-$. Then extend ω by 0 outside U:

$$\tilde{\omega}(p) = \begin{cases} \omega(p) & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

Then $\int_M d\omega = \int_U d\omega = \int_{\mathbb{R}^n_-} d\tilde{\omega} = \int_{\partial \mathbb{R}^n_-} \tilde{\omega} = \int_{\partial U} \omega$.

Proof of Stokes' theorem, general case

Our goal: find orientation-preserving charts (U_i, h_i) for $i = 1, \ldots, k$ and $\omega_i \in \Omega^{n-1}M$ such that

- $\omega = \omega_1 + \cdots + \omega_k$
- $\mathrm{supp}(\omega_i) \subset U_i$, with $\mathrm{supp}(\omega_i)$ compact.

It will then follow from our previous cases that

$$\int_M d\omega = \sum_{i=1}^{k} \int_M d\omega_i = \sum_{i=1}^{k} \int_{U_i} d\omega_i$$

$$= \sum_{i=1}^{k} \int_{\partial U_i} \omega_i = \sum_{i=1}^{k} \int_{\partial M} \omega_i$$

$$= \int_{\partial M} \sum_{i=1}^{k} \omega_i = \int_{\partial M} \omega.$$

Partition of unity

- For each $p \in \mathrm{supp}(\omega)$, choose an orientation-preserving chart (U_p, h_p). Then find a smooth function $\lambda_p \colon M \to [0,1]$ such that $\lambda_p(p) > 0$, with $\mathrm{supp}(\lambda_p)$ compact and contained in U_p (bump function).
- Let $\tilde{U}_p := \lambda_p^{-1}((0,1]) = M \setminus \lambda_p^{-1}(0)$. Then, $\tilde{U}_p$ is open, and $\mathrm{supp}(\omega) \subset \cup_{p \in M} \tilde{U}_p$.
- Since $\mathrm{supp}(\omega)$ is compact, there exist $p_1, \dots, p_k$ such that $\mathrm{supp}(\omega) \subset \cup_{i=1}^k \tilde{U}_{p_i} =: X$.
- For $i = 1, \dots, k$ define $\tau_i \colon X \to [0,1]$ by

$$\tau_i(x) = \frac{\lambda_{p_i}(x)}{\sum_{j=1}^k \lambda_{p_j}(x)} \in [0,1].$$

- Then $\tau_i \colon X \to [0,1]$, $\mathrm{supp}(\tau_i) \subset U_{p_i}$ is compact, and $\sum_{i=1}^k \tau_i(x) = 1$ for all $x \in X$.

Partition of unity

$$\tau_i(x) = \frac{\lambda_{p_i}(x)}{\sum_{j=1}^{k} \lambda_{p_j}(x)} \in [0,1]$$

$\tau_i \colon X \to [0,1]$, $\mathrm{supp}(\tau_i) \subset U_{p_i}$ is compact, and $\sum_{i=1}^{k} \tau_i(x) = 1$ for all $x \in X$.

Definition. For $i = 1, \ldots, k$, let

$$\omega_i(p) = \begin{cases} \tau_i(p)\omega(p) & \text{for } p \in X \\ 0 & \text{for } p \in M \setminus X. \end{cases}$$

Then

- $\omega = \omega_1 + \cdots + \omega_k$
- $\mathrm{supp}(\omega_i) \subset U_i$, with $\mathrm{supp}(\omega_i)$ compact.

Example

Let

$$\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy.$$

What is $\int_{S^2} \omega$, where $S^2 \subset \mathbb{R}^3$ is the 2-sphere?

Solution:

$$\int_{S^2} \omega = \int_{\partial D^3} \omega = \int_{D^3} d\omega = \int_{D^3} 3\, dx \wedge dy \wedge dz$$

$$= \int_{D^3} 3 = 3\,\mathrm{vol}(D^3) = 3 \cdot \frac{4}{3}\pi = 4\pi.$$