de Rham cohomology

Wednesday quiz

Let M be an oriented n-manifold, and let $\omega \in \Omega^n M$. What does it mean to say ω is *integrable*, and if it is integrable, what is the definition of $\int_M \omega$?

Functors

I.

$$\begin{array}{rcl} \text{Manifolds} & \to & \text{graded, differential, anticommutative, } \mathbb{R}\text{-algebras} \\ M & \mapsto & \Omega^\bullet M := \bigoplus_{k \geq 0} \Omega^k M \end{array}$$

II.

$$\begin{array}{rcl} \text{Manifolds} & \to & \text{chain complexes} \\ M & \mapsto & 0 \xrightarrow{d} \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \to \cdots \end{array}$$

(De Rham complex)

Contravariant

$f\colon M \to N$ induces a mapping of cochain complexes:

$$\begin{array}{ccccccccc}
0 & \xrightarrow{d} & \Omega^0 N & \xrightarrow{d} & \Omega^1 N & \xrightarrow{d} & \Omega^2 N & \xrightarrow{d} & \dots \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & \\
0 & \xrightarrow{d} & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \Omega^2 M & \xrightarrow{d} & \dots
\end{array}$$

Shorthand:

$$f^*\colon \Omega^\bullet N \to \Omega^\bullet M.$$

The diagram commutes: $f^* d\omega = df^*\omega$.

k-th de Rham cohomology group

$$\cdots \to \Omega^{k-1}M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1} \to \cdots$$

complex: $d^2 = 0$, i.e., $d_k \circ d_{k-1} = 0$ for all k; $\quad \operatorname{im} d_{k-1} \subseteq \ker d_k$.

closed forms = cocycles

$$H^k M := \frac{\ker\left(\Omega^k M \xrightarrow{d_k} \Omega^{k+1} M\right)}{\operatorname{im}\left(\Omega^{k-1} M \xrightarrow{d_{k-1}} \Omega^k M\right)}$$

exact forms = coboundaries

$$:= \quad \ker d_k / \operatorname{im} d_{k-1}$$

Cohomology

Suppose ω and η are k-cocycles. In other words $d_k\omega = d_k\eta = 0$. Denote their classes in $H^k M$ by $[\omega]$ and $[\eta]$.

Cocyles ω and η are *cohomologous* if $[\omega] = [\eta]$; in other words,

$$\omega = \eta + d\nu$$

for some $\nu \in \Omega^{k-1} M$.

Exactness

A sequence of linear mappings $U \xrightarrow{f} V \xrightarrow{g} W$ of vector spaces is *exact* at V if $\operatorname{im}(f) = \ker(g)$. Equivalently, $\ker(g)/\operatorname{im}(f) = 0$.

In general, if $\operatorname{im}(g) \subseteq \ker(f)$, then $\ker(g)/\operatorname{im}(f)$ measures how close the sequence is to being exact.

Cohomology of $M = \mathbb{R}$

$H^0\mathbb{R}$

$$0 \xrightarrow{d_{-1}} \Omega^0\mathbb{R} \xrightarrow{d_0} \Omega^1\mathbb{R} \xrightarrow{d_1} 0 \to \cdots$$

$$\Omega^0\mathbb{R} = \{\text{functions } f : \mathbb{R} \to \mathbb{R}\}, \qquad d_0 f = f'(x)\, dx.$$

$\ker d_0 =$ constant functions $\mathbb{R} \to \mathbb{R}$.

$$H^0\mathbb{R} = \ker d_0 / \operatorname{im} d_{-1} = \text{constant functions}/0 = \text{constant functions}.$$

$$H^0\mathbb{R} = \ker d_0 / \operatorname{im} d_{-1} = \text{constant functions} \approx \mathbb{R}$$

$$f \mapsto f(0)$$

Thus, $H^0\mathbb{R} \approx \mathbb{R}$.

Cohomology of $M = \mathbb{R}$

$H^1\mathbb{R}$

$$0 \xrightarrow{d_{-1}} \Omega^0\mathbb{R} \xrightarrow{d_0} \Omega^1\mathbb{R} \xrightarrow{d_1} 0 \to \cdots$$

$$\Omega^1\mathbb{R} = \{f(x)\,dx\}, \quad d_1(f(x)\,dx) = 0.$$

$$\ker d_1 = \Omega^1\mathbb{R}, \quad \operatorname{im} d_0 = \{g'(x)\,dx : g\colon \mathbb{R} \to \mathbb{R}\}$$

$$H^1\mathbb{R} = \ker d_1 / \operatorname{im} d_0 = \{f(x)\,dx\}/\{g'(x)\,dx\} = 0.$$

Reason: FTC, the fundamental theorem of calculus.

Given a function f, define $g(x) = \int_0^x f(t)\,dt$. Then $g' = f$.

Thus, $H^1\mathbb{R} = 0$. In sum, $H^0\mathbb{R} \approx \mathbb{R}$, and all other cohomology for $\mathbb{R}$ is 0. What about $\mathbb{R}^n$? S^n?, etc.

Algebraic structure

$H^\bullet M := \bigoplus_{k\geq 0} H^k M$ has a ring structure!

$$\begin{aligned} H^k M \times H^\ell M &\to H^{k+\ell} M \\ ([\omega],[\eta]) &\mapsto [\omega \wedge \eta]. \end{aligned}$$

Amazing fact: this mapping is well-defined.

Assignment: read the proof in our text.

New contravariant functor

$f\colon M \to N$ induces

$$\begin{aligned} f^{\#,k}\colon H^k N &\to H^k M \\ [\omega] &\mapsto [f^*\omega]. \end{aligned}$$

Thus, we have a mapping $f^{\#}\colon H^\bullet N \to H^\bullet M$.

Functor:

$$\begin{aligned} \text{Manifolds} &\quad\to\quad \text{graded anticommutative } \mathbb{R}\text{-algebras} \\ M &\quad\mapsto\quad H^\bullet M = \bigoplus_{k\geq 0} H^k M. \end{aligned}$$

First observations

1. $H^k M = 0$ if $k >$ dim M.
2. $H^0 M \approx \mathbb{R}^{\#\text{ components of } M}$
3. If M is a *closed* (i.e., compact with $\partial M = \emptyset$) and orientable n-manifold, then $H^n M \neq 0$.

   Proof. Pick $\omega \in \Omega^n M$ such that $\int_M \omega \neq 0$. (Why is this possible?) Claim: $\omega \neq d\eta$. Otherwise,

   $$0 \neq \int_M \omega = \int_M d\eta \underbrace{=}_{Stokes'} \int_{\partial M} \eta = \int_{\emptyset} \eta = 0.$$

   (The last step follows since the measure of the empty set is 0.) □