# Integration II

- Review definition of $\int_M \omega$.
- Review change of variables formula for $\mathbb{R}^n$.
- Formula for pulling back forms.
- Sketch proof that $\int_M \omega$ is independent of the choice of orienting atlas $\mathcal{A}$ and the measurable sets $A_i$. See our text, Theorem 10.4.
- Sufficient condition for integrability.
- Change of variables formula for manifolds.

# Warm-up

Let

$$\begin{aligned} f\colon (0,1) &\to \mathbb{R}^2 \\ t &\mapsto (t, t^2), \end{aligned}$$

and let $\omega = x\,dy$.

Compute $\int_{(0,1)} f^*\omega$.

$f^*\omega = t\,dt^2 = t \cdot 2t\,dt = 2t^2\,dt.$

$\int_{(0,1)} 2t^2\,dt = \int_{(0,1)} 2t^2 = \frac{2}{3}.$

# Change of variables formula

$K \subset \mathbb{R}^n$ compact, connected, $\mathrm{vol}(\partial K) = 0$, open $U$ with $K \subseteq U$.

$\phi$: $U \to \mathbb{R}^n$ a $C^1$ mapping, injective on the interior $K^\circ$, with $\det(J\phi) \neq 0$ on $K^\circ$

$f : \phi(K) \to \mathbb{R}$ continuous.

$$K \xrightarrow{\phi} \phi(K) \xrightarrow{f} \mathbb{R}$$

**Theorem (change of variables).**

$$\int_{\phi(K)} f = \int_K (f \circ \phi) |\det(J\phi)|.$$

# Local formula for pulling back forms

Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$, with coordinates $x_1, \dots, x_n$ on the domain and $y_1, \dots, y_n$ on the codomain.

Let $\omega = a(y)\, dy_1 \wedge \cdots \wedge dy_n \in \Omega^n \mathbb{R}^n$.

Then

$$
\begin{aligned}
(f^*\omega)(x) &= a(f(x))\, df_1 \wedge \cdots \wedge df_n \\
&= (a \circ f)(x) \, \left( \frac{\partial f_1}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f_1}{\partial x_n}\, dx_n \right) \wedge \\
&\qquad\qquad \cdots \wedge \left( \frac{\partial f_n}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f_n}{\partial x_n}\, dx_n \right) \\
&= (a \circ f)(x) \det(Jf(x))\, dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}
$$

## $\int_M \omega$ is independent of choices

**Theorem.** $\int_M \omega$ is independent of the choice of $\mathfrak{A}$ and the $A_i$.

**Proof.** See page 50 of our notes.

# A sufficient condition for integrability

The *support* of $\omega \in \Omega^n M$ is

$$\mathrm{supp}(\omega) = \overline{\{p \in M \colon \omega(p) \neq 0\}}.$$

**Proposition.** If $\mathrm{supp}(\omega)$ is compact, then $\int_M \omega$ exists.

Idea of proof: the sums involved can be taken to be finite.

**Corollary.** If $M$ is compact, the $\int_M \omega$ exists.

# Change of variables for a manifold

**Proposition.** Let $f\colon M \to N$ be an orientation preserving diffeomorphism of oriented $n$-manifolds, and let $\omega \in \Omega^n N$. Then

$$\int_M f^*\omega = \int_N \omega.$$

**Idea:** We may use the diffeomorphism $f$ to choose compatible orienting atlases for $M$ and $N$ and compatible decompositions into measurable sets.

Then the question is local. So we may assume $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\omega = a(y)\, dy_1 \wedge \cdots \wedge dy_n$. So

$$\begin{aligned}
\int_M f^*(a\, dy_1 \wedge \cdots \wedge dy_n) &= \int_M (a \circ f) \det(Jf)\, dx_1 \wedge \cdots \wedge dx_n \\
&= \int_{f(M)} a\, dy_1 \wedge \cdots \wedge dy_n \\
&= \int_N \omega.
\end{aligned}$$