Orientations

Quiz

Describe the construction of the tangent bundle $\pi\colon TM \to M$ on a manifold M.

- What are the fibers, $\pi^{-1}(p)$?
- What are the charts for TM?
- If (U, h) and (V, k) are charts on M, what is the transition function from $\pi^{-1}(U)$ to $\pi^{-1}(V)$ on TM?

Reminder

Don't forget to advertise the math/stat colloquium!

Orientation of a vector space

Let $\mathcal{B}$: $v_1, \ldots, v_n$ and $\mathcal{C}$: $w_1, \ldots, w_n$ be ordered bases for a vector space V.

Define a linear function $\phi\colon V \to V$ by $\phi(v_i) = w_i$ for all i.

Say $\mathcal{B} \sim \mathcal{C}$ if $\det \phi > 0$. In this case, we say $\mathcal{B}$ and $\mathcal{C}$ have the same orientation.

The relation $\sim$ is an equivalence relation on the set of ordered bases for V.

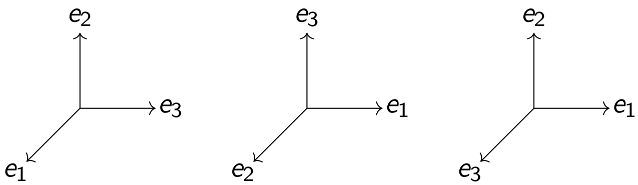
An equivalence class for $\sim$ is an *orientation* of V.

The *positive* orientation on $\mathbb{R}^n$ is the equivalence class of the standard ordered basis.

Two interesting properties of the determinant of a matrix A: $|\det A|$ gives volume and $\mathrm{sign}(A)$ gives the *orientation* of the rows (or columns) of A.

Orientation of a vector space

Which of the following have the same orientation?



$\langle e_1, e_2, e_3 \rangle$ $\quad$ $\langle e_2, e_3, e_1 \rangle$ $\quad$ $\langle e_3, e_2, e_1 \rangle$

Orientation of a manifold

For each $p \in M$, let $\mathcal{O}_p$ be an orientation if T_pM.

A collection of orientations $\mathcal{O} = \{\mathcal{O}_p\}_{p \in M}$ is locally coherent if for all $p \in M$, there exists a chart (U, h) at p such that the isomorphism $T_qM \xrightarrow{\sim} \mathbb{R}^n$ induced by (U, h) sends $\mathcal{O}_q$ to the positive orientation of $\mathbb{R}^n$ for all $q \in U$.

Recall the local charts on TM:

$$\begin{aligned} \amalg_{p \in U} T_pM = \pi^{-1}(U) &\xrightarrow{\sim} U \times \mathbb{R}^n \\ v \in T_pM &\mapsto (p, v(U, h)) \end{aligned}$$

Restricted to the fiber $\pi^{-1}(p)$:

$$\begin{aligned} T_pM &\xrightarrow{\sim} \{p\} \times \mathbb{R}^n &&\simeq \mathbb{R}^n \\ v &\mapsto (p, v(U, h)) &&\mapsto v(U, h). \end{aligned}$$

Oriented manifold

An *oriented manifold* is a pair $(M, \mathcal{O})$ where $\mathcal{O} = \{\mathcal{O}_p\}_{p \in M}$ is a locally coherent collection of orientations $\mathcal{O}_p$ for each tangent space $T_p M$.

Take chart (U, h) at p required by local coherence, and take chart (V, k) at q required by local coherence. Suppose that $s \in U \cap V$. Is there a possible contradiction? No.

$$\mathbb{R}^n \xrightarrow[\sim]{\mathcal{O}_s} T_s M \xrightarrow[\sim]{dh_s} \mathbb{R}^n \xleftarrow[\sim]{dk_s} T_s M \xleftarrow[\sim]{\mathcal{O}_s} \mathbb{R}^n.$$

Orientation preserving diffeomorphism

Suppose M and N are orientated manifolds. Then a diffeomorphism $f\colon M \to N$ is orientation preserving if $df_p\colon T_pM \to T_{f(p)}N$ is orientation preserving for all $p \in M$.

This means that for all $p \in M$, applying df_p to a representative for $\mathcal{O}_p$ on M gives an ordered basis in $\mathcal{O}_{f(p)}$ on N.

Orienting atlas

An atlas $\mathfrak{A} = \{(U_i, h_i)\}_{i \in I}$ is orienting if $\det J(h_j \circ h_i^{-1}) > 0$ for all $i, j \in I$.

An orienting atlas induces an orientation on M: for each $(U_i, h_i) \in \mathfrak{A}$, orient T_pM using

$$\begin{aligned} T_pM &\xrightarrow{\sim} \mathbb{R}^n \\ v &\mapsto v(U_i, h_j). \end{aligned}$$

Choose the ordered basis on T_pM that is sent to the standard basis for $\mathbb{R}^n$.

Compatibility follows from the fact that $\det J(h_j \circ h_i^{-1}) > 0$.

Question. Does every orientable manifold have an orienting atlas? (See next page.)

Question. Does every orientable manifold have an orienting atlas?

Answer: Yes. If M is orientable, the it has a locally coherent collection of orientations $\mathcal{O} = \{\mathcal{O}_p\}_{p \in M}$. See the definition of *local coherence*. It guarantees the existence of certain charts, and these charts will form an orienting atlas.

$$
\begin{tikzcd}
 & & v_1, \ldots, v_n \in \mathcal{O}_p \arrow[ddll, dashed, bend right] \arrow[ddrr, dashed, bend left] & & \\
 & & T_pM \arrow[dl, bend right] \arrow[dr, bend left] & & \\
 & \mathbb{R}^n \arrow[rr, "J(k \circ h^{-1})(p)"] & & \mathbb{R}^n & \\
v_1(U,h), \ldots, v_n(U,h) & & & & v_1(V,k), \ldots, v_n(V,k)
\end{tikzcd}
$$

Both bases having positive orientation implies $\det J(k \circ h^{-1}) > 0$.

Claim. Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ be positively oriented bases of $\mathbb{R}^n$, and let J be an $n \times n$ matrix such that $\mathbb{R}^n \xrightarrow{J} \mathbb{R}^n$ sends u_i to w_i for all i. Then $\det(J) > 0$.

Proof. Let A, resp. B, be the matrices with the u_i, resp. w_i, as columns.

We have the commutative diagram

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dl, "A"'] \arrow[dr, "B"] & \\ \mathbb{R}^n \arrow[rr, "J"] & & \mathbb{R}^n \end{tikzcd}$$

Then, $JA = B \Rightarrow \det(JA) = \det(B) \Rightarrow \det(J)\det(A) = \det(B)$. Etc. $\square$

Proposition. A manifold with an atlas consisting of a single chart is orientable.

Proposition. If M has an atlas consisting of two charts (U, h) and (V, k), then M is orientable.

Proof. If $\det J(k \circ h^{-1}) < 0$, then replace $k = (k_1, \ldots, k_n)$ by $(-k_1, k_2, \ldots, k_n)$. (Compose k with the linear mapping sending e_1 to $-e_1$ and fixing the other standard basis vectors. That will change the sign of the determinant.)

Example. The n-sphere, S^n.

Example. What about the Möbius strip? What's wrong with the statement of the Proposition?

We need to assume that $U \cap V$ is connected. Why?

Theorem. Let $n = \dim M$. Then M is orientable if and only if it has a nonvanishing n-form $\omega \in \Omega^n M$.

Sketch of proof. ($\Leftarrow$) Say $\omega \in \Omega^n M$ is nonvanishing. Let $\mathfrak{A}$ be any atlas. We may assume that for all $(U, h) \in \mathfrak{A}$, we have the corresponding local form $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ with $f > 0$ on U. (Why? If not, change h by negating its first component, as before.)

Claim: the resulting atlas is orienting. Idea: suppose $\omega = g\, dy_1 \wedge \cdots \wedge dy_n$ with respect to another chart (V, k). How does the local form of ω change?

Answer: $g\, dy_1 \wedge \cdots \wedge dy_n$ will pull back to

$$\det(J(h \circ k^{-1})^t) g(h \circ k^{-1})\, dx_1 \wedge \cdots dx_n = f\, dx_1 \wedge \cdots \wedge dx_n.$$

Since $f > 0$ and $g > 0$, it follows that $\det J(k \circ h^{-1}) > 0$.