Exterior derivative

De Rham Complex

Theorem 8.5 Let M be a manifold of dimension n. There exists a unique sequence of linear maps

$$0 \to \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \Omega^3 M \xrightarrow{d} \cdots \qquad (1)$$

such that:

(a) If $f \in \Omega^0 M$, i.e., $f\colon M \to \mathbb{R}$, then df is the normal differential;

(b) Sequence (1) is a complex, i.e., $d^2 = d \circ d = 0$;

(c) d satisfies the product rule: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ for $\omega \in \Omega^r M$.

In local coordinates, d is given by

$$d\left(\sum_\mu a_\mu dx_\mu\right) = \sum_\mu \sum_{i=1}^n \frac{\partial a_\mu}{\partial x_i} dx_i \wedge dx_\mu.$$

Exterior derivative

Taking local coordinates, suppose that $M = \mathbb{R}^n$, and let $f\colon M = \mathbb{R}^n \to \mathbb{R}$. Then,

$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n$$

$$\begin{aligned} d(f\, dx_\mu) &= df \wedge dx_\mu \\ &= \left(\frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n \right) \wedge dx_\mu \\ &= \frac{\partial f}{\partial x_1}\, dx_1 \wedge dx_\mu + \cdots + \frac{\partial f}{\partial x_n}\, dx_n \wedge dx_\mu . \end{aligned}$$

Example. $d(yz^2\, dx) =$

$d^2(yz^2\, dx) =$

Cochains

$$\Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \cdots$$

$$d^2 = 0$$

See the Wikipedia article on chain complexes.

Mappings of cochains

$f: M \to N$ induces a mapping of cochain complexes:

$$\begin{array}{ccccccccc} 0 & \xrightarrow{d} & \Omega^0 N & \xrightarrow{d} & \Omega^1 N & \xrightarrow{d} & \Omega^2 N & \xrightarrow{d} & \dots \\ {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow & & \\ 0 & \xrightarrow{d} & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \Omega^2 M & \xrightarrow{d} & \dots \end{array}$$

Shorthand:

$$f^*: \Omega^\bullet N \to \Omega^\bullet M.$$

The diagram commutes: $f^* d\omega = d f^* \omega$.

Mappings of cochains

Check that $f^*\colon \Omega^\bullet N \to \Omega^\bullet M$ in the case $\omega \in \Omega^0 N$, i.e., when $\omega\colon N \to \mathbb{R}$.

Taking local coordinates, we may assume $M = \mathbb{R}^m$ with coordinates $x_1, \ldots, x_n$, and $N = \mathbb{R}^n$ with coordinates $y_1, \ldots, y_n$.

Compute $f^* d\omega$ and $df^*\omega$, then compare. (What comes to the rescue?)

Cohomology

$$\cdots \to \Omega^{k-1}M \xrightarrow{d_{k-1}} \Omega^k M \xrightarrow{d_k} \Omega^{k+1}M \to \cdots$$

$d^2 = 0$, i.e. $d_k \circ d_{k-1} = 0 \quad \Rightarrow \quad \operatorname{im} d_{k-1} \subseteq \ker d_k$

k-th *De Rham cohomology* group of M:

$$H^k M := \ker d_k / \operatorname{im} d_{k-1}.$$

k-th *Betti number* of M:

$$\beta_i(M) := \dim_{\mathbb{R}} H_k M.$$

See the Wikipedia article on De Rham cohomology.