Integration I

Lebesgue measurable sets in $\mathbb{R}^n$

Definition C.6. If $I \in \mathbb{R}^n$ is a rectangle, define $\mu(I)$ to be the product of the lengths of its sides. The *outer measure* of $X \subset \mathbb{R}^n$ is

$$\mathrm{outer}(X) = \inf \left\{ \sum_{i=1}^{\infty} \mu(I_i) \right\}$$

where the inf is over all sequences of rectangles $I_1, I_2, \ldots$ covering X, i.e., such that $X \subseteq \cup_{i=1}^{\infty} I_i$.

The set X is *Lebesgue measurable* if it "splits additively in measure":

$$\mathrm{outer}(X) = \mathrm{outer}(X \cap A) + \mathrm{outer}(X \cap A^c)$$

for all $A \subseteq \mathbb{R}^n$.

Lebesgue measurable sets in $\mathbb{R}^n$

Theorem C.7. The collection of Lebesgue measurable sets $\mathcal{L}$ in $\mathbb{R}^n$ forms a σ-algebra.

It contains all open sets (and, hence, all closed sets).

Outer measure restricted to $\mathcal{L}$ forms is a measure, i.e., defining $\mu(A) = \mathrm{outer}(A)$ for all $A \in \mathcal{L}$, it follows that $(\mathbb{R}^n, \mathcal{L}, \mu)$ is a measure space.

Examples of sets of measure 0 in $\mathbb{R}^2$.

- A single point.
- The integers on the x-axis.
- A circle.

Warning: measure 0 is not the same as unmeasureable.

Measurable sets in a manifold

The set $A \subseteq M$ is *measurable* if $h(A \cap U) \subseteq \mathbb{R}^n$ is measurable for all charts (U, h).

Integration of n-forms

M an oriented n-manifold, $\qquad \omega \in \Omega^n M$

Sketch of definition of $\int_M \omega$:

- Pick an orienting atlas.
- Partition M measurable pieces A_i, each A_i contained in a chart.
- Locally, on A_i, we have $\omega = \tilde{a}_i(x)\, dx_1 \wedge \cdots \wedge dx_n$.
- Define $\int_{A_i} \omega|_{A_i} = \int_{h(A_i)} a_i(x)\, dx_1 \wedge \cdots \wedge dx_n = \int_{h(A_i)} a_i$ where $a_i = \tilde{a}_i \circ h^{-1}$.
- Define
$$\int \omega = \int_M \omega = \sum_i \int_{A_i} \omega|_{A_i}$$

Notes: We have tacitly assumed there are a countable number of A_i.

Partitioning M into the A_i

- Let $\{(V_\alpha, k_\alpha)\}_\alpha$ be an orienting atlas. We will create a new orienting atlas $\mathfrak{A}$ and the sets A_i.
- Let $\mathcal{B}$ be a countable basis for the topology of M. (We are assuming M is *second countable*.)
- For all $p \in M$, find an element $U \in \mathcal{B}$ such that $p \in U \subseteq V_\alpha$ for some α. Let $h = k_\alpha|_U$ and add (U, h) to $\mathfrak{A}$.
- Since $\mathcal{B}$ is countable, we can write $\mathfrak{A} = \{(U_i, h_i)\}_{i=1,2,\ldots}$.
- Let $A_1 := U_1$. For $i \geq 1$, let $A_{i+1} := U_{i+1} \setminus \left(\cup_{j=1}^{i} A_i\right)$.

Integration

Definition of $\int_M \omega$. Choose a countable orienting atlas $\mathfrak{A} = \{(U_i, h_i)\}_i$ and a collection of measurable sets A_i such that the A_i partition M and $A_i \subseteq U_i$ for all i.

On (U_i, h_i), we have $\omega(p) = \tilde{a}_i(p)\, dx_{1,p} \wedge \cdots \wedge dx_{n,p}$ where $\tilde{a}_i \colon U_i \to \mathbb{R}$. Define $a_i = \tilde{a}_i \circ h_i^{-1}$.

Then ω is *integrable* if each $a_i \colon h_i(U_i) \to \mathbb{R}$ is integrable on $h(A_i)$ (automatic, since ω is smooth) and $\sum_i \int_{h(A_i)} |a_i| < \infty$.

In this case, we define the integral to be the sum:

$$\int_M \omega = \sum_i \int_{h_i(A_i)} a_i.$$

Theorem. $\int_M \omega$ is independent of the choice of $\mathfrak{A}$ and the A_i.