Tangent bundle

Quiz

1. Let (U, h) be a chart on a manifold M. What is the meaning of the i-th standard basis vector $(\partial/\partial x_i)_p$ with respect to (U, h) as an element of $T_p^{\mathrm{alg}}M$ (derivations on germs at p)? In other words, if f is a germ at p, what is $(\partial/\partial x_i)_p(f)$?

2. Let $f\colon M \to N$ be a mapping of manifolds of dimensions m and n, respectively, and let $p \in M$. Consider the corresponding differential mapping $df_p\colon T_pM \to T_{f(p)}N$ where we interpret the tangent spaces using their "physical" interpretations, as mapping of charts to $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Let $v \in T_p^{\mathrm{phy}}M$, and let (V, k) be a chart on N at $f(p)$. Then $df_p(v) \in T_{f(p)}^{\mathrm{phy}}N$. How would you find $df_p(v)(V, k)$?

Tangent bundle

$$\begin{tikzcd}
TM = \amalg_p T_pM \arrow[d, "\pi"'] & v \in T_pM \arrow[d, mapsto] \\
M & p
\end{tikzcd}$$

Tangent bundle: manifold structure

Chart at $p \in M$: $\quad (U, h)$

Corresponding chart on TM:

$$\begin{aligned} \pi^{-1}(U) &\xrightarrow{\tilde{h}} h(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \\ v \in T_q M &\mapsto (h(q), v(U, h)), \end{aligned}$$

taking $T_q M = T_q^{\mathrm{phy}} M$, for instance.

Tangent bundle: manifold structure

$$v \in T_q M \longmapsto (h(q), v(U,h))$$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{h}} & h(U) \times \mathbb{R}^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{h} & h(U) \end{array}$$

$$q \longmapsto h(q)$$

Tangent bundle: transition functions

Take charts (U, h) and (V, k) at $p \in M$.

$$w \in T_q M$$

$$\begin{tikzcd}[column sep=tiny]
& \pi^{-1}(U) \cap \pi^{-1}(V) \arrow[dl] \arrow[dr] & \\
h(U \cap V) \times \mathbb{R}^n \arrow[rr, dashed] & & k(U \cap V) \times \mathbb{R}^n
\end{tikzcd}$$

$$(h(q), w(U, h)) \longmapsto (k(q), w(V, k))$$

The dashed map is $(k \circ h^{-1}, D(k \circ h^{-1}))$. We are using the fact that $D_{h(q)}(k \circ h^{-1}) w(U, h) = w(V, k)$.

So TM is a manifold. Is $TM \to M$ a vector bundle?

General vector bundle $\pi\colon E \to M$ of rank k

Conditions:

(1) For all $p \in M$, the fiber $E_p := \pi^{-1}(p)$ is a k-dimensional vector space.

(2) Locally trivial: for all $p \in M$, there exists a neighborhood U of p and a diffeomorphism ϕ_U such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\
& U &
\end{tikzcd}$$

(3) $\phi_U\colon E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

TM is a vector bundle

Locally trivial:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

$\phi_U = (h^{-1} \times \mathrm{id}) \circ \tilde{h}$

Summary of construction of TM

1. Definition as a set: $TM = \amalg_{p \in M} T_p M$.
2. Manifold structure: $\pi^{-1}(U) \xrightarrow{\tilde{h}} h(U) \times \mathbb{R}^n$.
3. Transition functions:

$$\begin{tikzcd} & \pi^{-1}(U) \cap \pi^{-1}(V) \arrow[dl] \arrow[dr] & \\ h(U \cap V) \times \mathbb{R}^n \arrow[rr, "h^{-1} \circ k \times D(h^{-1} \circ k)"] & & k(U \cap V) \times \mathbb{R}^n \end{tikzcd}$$

4. Vector bundle properties: locally trivial with fibers linearly isomorphic to $\mathbb{R}^n$.

Tangent mapping

A mapping of manifolds $f\colon M \to N$, induces a mapping of tangent bundles:

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Locally:

$$\begin{array}{ccccccc} \pi^{-1}(U) & \xrightarrow{\simeq} & h(U)\times\mathbb{R}^m & \xrightarrow{\tilde{f}\times D\tilde{f}} & k(V)\times\mathbb{R}^n & \xleftarrow{\simeq} & \pi^{-1}(V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{h} & h(U) & \xrightarrow{\tilde{f}} & k(V) & \xleftarrow{k} & V \end{array}$$

where $\tilde{f} := k \circ f \circ h^{-1}$.