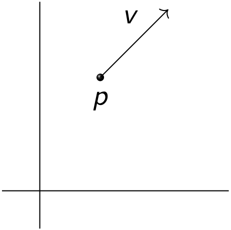
Vector bundles

Miscellaneous

- The quiz material for Wednesday is posted.
- Reading for today: Section 6.
- Reading for tomorrow: Section 7.

Tangent bundle on $\mathbb{R}^n$

$$
\begin{tikzcd}
(p, v) \arrow[d, mapsto] & \mathbb{R}^n \times \mathbb{R}^n \arrow[d, "\pi"] \\
p & \mathbb{R}^n
\end{tikzcd}
$$



fiber at p: $\pi^{-1}(p) = \{p\} \times \mathbb{R}^n \simeq \mathbb{R}^n =$ vector space

For example, $2(\{p\} \times u) \;+\; 3(\{p\} \times v) = \{p\} \times (2u + 3v)$.

Section of tangent bundle

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^n \arrow[d, "\pi"'] & (p, v) \\ \mathbb{R}^n \arrow[u, bend right, "s"'] & p \arrow[u, mapsto] \end{tikzcd}$$

$$\pi \circ s = \mathrm{id}_{\mathbb{R}^n}$$

Question: Why can we identify a section with a "flow" in $\mathbb{R}^n$?
So a section of the tangent bundle is a vector field.

Trivial bundle of rank k

$$\begin{array}{c} M \times \mathbb{R}^k \\ \Big\downarrow \pi \\ M \end{array}$$

$$\pi(m, v) = m$$

Charts

M: (U, h)

$M \times \mathbb{R}^k$: $(U \times \mathbb{R}^k, h \times \mathrm{id})$

Trivial bundle of rank k

$$\begin{tikzcd} M \times \mathbb{R}^k \arrow[d, "\pi"] \\ M \end{tikzcd}$$

Charts

M: (U, h)

$M \times \mathbb{R}^k$: $(U \times \mathbb{R}^k, h \times \mathrm{id})$

$$\pi(m, v) = v$$

$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r, "h \times \mathrm{id}"] \arrow[d, "\pi"] & h(U) \times \mathbb{R}^k \arrow[d, "\pi"] & \subseteq \mathbb{R}^n \times \mathbb{R}^k \arrow[d, "\pi"] \\ U \arrow[r, "h"] & h(U) & \subseteq \mathbb{R}^n \end{tikzcd}$$

General vector bundle on M of rank k

$$
\begin{array}{lcr}
\text{total space} & E & \text{manifold of dimension } n+k \\
 & \Big\downarrow \pi & \pi \text{ is smooth} \\
\text{base space} & M & \text{manifold of dimension } n
\end{array}
$$

(Conditions appear on next slide.)

General vector bundle $\pi\colon E \to M$ of rank k

Conditions:

(1) For all $p \in M$, the fiber $E_p := \pi^{-1}(p)$ is a k-dimensional vector space.

(2) Locally trivial: for all $p \in M$, there exists a neighborhood U of p and a diffeomorphism ϕ_U such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\ & U & \end{tikzcd}$$

(3) $\phi_U\colon E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

Sections

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} \pi^{-1}(U) \arrow[d, "\pi"'] \\ U \arrow[u, bend right, "s"'] \end{tikzcd}$$

$$\pi \circ s = \mathrm{id}_M \qquad \pi \circ s = \mathrm{id}_U$$

global section — local section

Questions: Do sections always exist? Do nonzero sections always exist?

Extending sections

Application of bump functions, on blackboard.

Motivation for tangent space

Formula for the length of a curve $\alpha\colon (a,b) \to \mathbb{R}^n$:

$$\begin{aligned}
\operatorname{len}(\alpha) &= \int_a^b |\alpha'(t)| \\
&= \int_a^b \sqrt{\alpha'(t) \cdot \alpha'(t)} \\
&= \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle} \\
&= \int_a^b \sqrt{\langle \alpha', \alpha' \rangle_t}
\end{aligned}$$

If we replace $\mathbb{R}^n$ with an arbitrary manifold M, where should $\langle\ ,\ \rangle_t$ live?

Tangent bundle

$$\begin{tikzcd} TM = \amalg_p T_pM \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

We will talk about the manifold structure of TM next time. We would like:

$\langle\ ,\ \rangle_p \colon T_pM \times T_pM \to \mathbb{R}$ is bilinear and symmetric.

Therefore, $\langle\ ,\ \rangle_p$"$\in$"$(\mathrm{Sym}^2\, T_pM)^* \simeq \mathrm{Sym}^2(T_p^*M)$.

We can also construct a vector bundle $\mathrm{Sym}^2(T^*M) \to M$. Then, $\langle\ ,\ \rangle \colon M \to \mathrm{Sym}^2(T^*M)$ should be a section, allowing us to find lengths of curves in M.