Differential Forms

Pullbacks of exterior products

$L\colon V \to W$ induces:

$$\begin{aligned} L^*\colon W^* &\to V^* \\ \phi &\mapsto \phi \circ L \end{aligned}$$

and

$$\begin{aligned} \Lambda^\ell L^*\colon \Lambda^\ell W^* &\to \Lambda^\ell V^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell &\mapsto L^*\phi_1 \wedge \cdots \wedge L^*\phi_\ell = (\phi_1 \circ L) \wedge \cdots \wedge (\phi_\ell \circ L). \end{aligned}$$

We will sometimes write L^* for $\Lambda^\ell L^*$ if the meaning is clear from context.

Standard basis for T_pM w.r.t. chart (U, h)

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$$

geometric: $(\partial/\partial x_i)_p = [\alpha]$ where $\alpha(t) = h^{-1}(h(p) + te_i)$

algebraic: $(\partial/\partial x_i)_p f = \frac{\partial f \circ h^{-1}}{\partial x_i}(p)$

physical: $(\partial/\partial x_i)_p(U, h) = e_i$

Bases with respect to (U, h)

T_pM: $\left(\frac{\partial}{\partial x_i}\right)_p$

T_p^*M: $dx_{i,p} := \left(\frac{\partial}{\partial x_i}\right)_p^*$, $dx_{i,p}(\left(\frac{\partial}{\partial x_j}\right)_p) = \delta(i,j)$

$\Lambda^\ell T_p^*M$: $dx_{\mu,p} := dx_{\mu_1,p} \wedge \cdots \wedge dx_{\mu_\ell,p}$

$$\mu \colon 1 \leq \mu_1 < \cdots < \mu_\ell \leq n$$

Example of an element in $\Lambda^2 T_p^*\mathbb{R}^3$:

$$4\, dx_p \wedge dy_p + 10\, dx_p \wedge dz_p - 7 dy_p \wedge dz_p$$

Review: Tangent bundle manifold structure

$$v \in T_qM \longmapsto (h(q), v(U,h))$$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{h}} & h(U) \times \mathbb{R}^n \\ \downarrow \pi & & \downarrow \pi \\ U & \xrightarrow{h} & h(U) \end{array}$$

$$q \longmapsto h(q)$$

Review: Tangent bundle manifold structure

$$\left(\frac{\partial}{\partial x_i}\right)_p \longmapsto (h(p), e_i)$$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{h}} & h(U) \times \mathbb{R}^n \\ \downarrow \pi & & \downarrow \pi \\ U & \xrightarrow{h} & h(U) \end{array}$$

$$p \longmapsto h(p)$$

Tangent bundle: transition functions

Take charts (U, h) and (V, k) at $p \in M$:

$$U \xrightarrow{\cong} h(U) \subseteq \mathbb{R}^n, \text{ with coords. } x_1, \ldots, x_n$$
$$V \xrightarrow{\cong} k(U) \subseteq \mathbb{R}^n, \text{ with coords. } y_1, \ldots, y_n$$

$$\begin{array}{ccc} \left(\frac{\partial}{\partial x_i}\right)_p & \pi^{-1}(U) \cap \pi^{-1}(V) & \left(\frac{\partial}{\partial y_i}\right)_p \\ & \swarrow \qquad \searrow & \\ h(U \cap V) \times \mathbb{R}^n & \xrightarrow{(k \circ h^{-1}, D(k \circ h^{-1}))} & k(U \cap V) \times \mathbb{R}^n \end{array}$$

$\left(\frac{\partial}{\partial x_j}\right)_p = \sum_{i=1}^n a_{ij} \left(\frac{\partial}{\partial y_i}\right)_p$ where $(a_{1j}, \ldots, a_{nj})^t$ is the j-th column of $D_p(k \circ h^{-1})$.

Exterior power of the cotangent bundle

$$\begin{array}{ccc} \Lambda^\ell T^*M = \amalg_p \Lambda^\ell T_p^* M & \qquad & v \in \Lambda^\ell T_p^* M \\ \Big\downarrow \pi & & \Big\downarrow \\ M & & p \end{array}$$

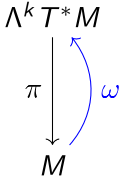
$$\pi^{-1}(U) \xrightarrow[\tilde{h}]{\simeq} h(U) \times \mathbb{R}^{\binom{n}{\ell}}$$

Exterior power of cotangent bundle: transition functions

$$\begin{array}{ccc} & \pi^{-1}(U)\cap\pi^{-1}(V) & \\ \swarrow & & \searrow \\ h(U\cap V)\times\mathbb{R}^{\binom{n}{\ell}} & \xleftarrow[(h\circ k^{-1},\, \Lambda^{\ell} D(k\circ h^{-1})^{*})]{} & k(U\cap V)\times\mathbb{R}^{\binom{n}{\ell}} \end{array}$$

Differential forms

A *differential k-form on M is a section of $\Lambda^k T^* M$:*



$$\pi \circ \omega = \mathrm{id}_M$$

Notation: $\Omega^k M$ is the $\mathbb{R}$-vector space k-forms.

Letting $C^\infty(M)$ be the ring of smooth functions $M \to \mathbb{R}$, we have that $\Omega^k M$ is a $C^\infty(M)$-module: if $f \in C^\infty(M)$ and $\omega \in \Omega^k M$, then

$$(f\omega)(p) = f(p)\omega(p) \in \Lambda^k T_p^* M.$$

k-forms in local coordinates

$$\begin{array}{ccc}
\coprod_{p \in U} \Lambda^k_p T^* M = \pi^{-1}(U) & \xrightarrow{\ \simeq\ } & U \times \Lambda^k(\mathbb{R}^n)^* \\
\Big\downarrow \pi & & \\
U & & \\
\Big\downarrow \simeq & & \\
\mathbb{R}^n \supseteq \ h(U) & & \\
x_1, \ldots, x_n & &
\end{array}$$

$$\omega(p) = \sum_{i=1}^{n} f_\mu(p)\, dx_{\mu,p}$$

$$f_\mu \colon U \to \mathbb{R}$$

Zero forms

$\Lambda^0 V := \mathbb{R}$

Claim: $\Omega^0 M = C^\infty(M)$

$$\begin{array}{ccc} \Lambda^0 T^*M & = \amalg_p \Lambda^0 T_p^* M = & \amalg_p \mathbb{R} \\ \downarrow \ \uparrow \omega & & \\ M & & \end{array}$$

So $\omega(p) \in \mathbb{R}$ for all $p \in M$.

Zero forms

$f\colon M \to \mathbb{R}$ induces

$$\begin{array}{ccc} TM & \xrightarrow{\quad df \quad} & T\mathbb{R} = \mathbb{R} \times \mathbb{R} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad f \quad} & \mathbb{R} \end{array}$$

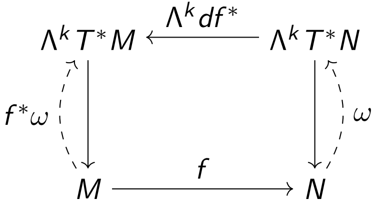
In coordinates (replacing T_pM by $\mathbb{R}^n$ and f by $f \circ h^{-1}$):

$$\begin{aligned} df_p\colon\ T_pM &\to \mathbb{R} \\ v &\mapsto \left[\frac{\partial f}{\partial x_1}(p) \cdots \frac{\partial f}{\partial x_n}(p)\right] v \end{aligned}$$

So $df_p \in T_p^*M$. $df_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, dx_i, p$. Check: what is $df_p(\frac{\partial}{\partial x_j})_p$?

Pullbacks of k-forms

$f\colon M \to N$ induces $TM \xrightarrow{df} TN$.



$$
\begin{aligned}
f^*\colon \Omega^k N &\to \Omega^k M \\
\omega &\mapsto \Lambda^k df^*(\omega \circ f)
\end{aligned}
$$

Pullbacks of k-forms in coordinates

Take coordinates $y_1, \ldots, y_n$ on N and $x_1, \ldots, x_m$ on M.

$f\colon M \to N$, locally, $f = (f_1, \ldots, f_n)$

$f^*\colon \Omega^k N \to \Omega^k M$

$\omega = \sum_\mu \omega_\mu(y_1, \ldots, y_n)\, dy_\mu$ where each ω_μ is a $\mathbb{R}$-valued function on N

To compute $f^*\omega$, substitute f_i for y_i:

$f^*\omega = \sum_\mu \omega_\mu(f_1(x), \ldots, f_n(x))\, df_{\mu_1} \wedge \cdots \wedge df_{\mu_k}$

Example

$$f : \mathbb{R}^3 \to \mathbb{R}^4$$
$$(x, y, z) \mapsto (x + 2y, z, xy + 1, y^2)$$

with coords p, q, r, s on $\mathbb{R}^4$

$\omega = p^2\, dp \wedge dr + r\, dq \wedge ds$

Compute $f^*\omega$.

Example

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \mapsto (x + 3y^2, x^2 + y^3 z)$$

with coords u, v on $\mathbb{R}^2$

$\omega = u\, du \wedge dv$

Compute $f^*\omega$.