Tensors and duality

Quiz

Let M be a manifold, and let $p \in M$.

1. Let $\mathcal{E}_p$ denote the vector space of germs of differentiable functions at p. What is $T_p^{\mathrm{alg}} M$?
2. Let $v \in T_p^{\mathrm{alg}} M$, and let $\mathcal{D}_p$ denote the set of charts at p. Describe the corresponding element of $T_p^{\mathrm{phy}} M$ under our mapping $T_p^{\mathrm{alg}} M \to T_p^{\mathrm{phy}} M$.

Vector space of linear functions

Let V and W be finite-dimensional vector spaces over $\mathbb{R}$.

Define

$$\hom(V, W) = \{\text{linear functions } V \to W\}$$

with linear structure

$$(\lambda f + g)(v) = \lambda f(v) + g(v)$$

for all $f, g \in \hom(V, W)$ and $\lambda \in \mathbb{R}$.

Dual space

$V^* := \hom(V, \mathbb{R})$

Example. Let $V = \mathbb{R}^3$. Then $\phi(x, y, z) = 3x - 27 + 4z$ defines an element $\phi \in V^*$.

Dual basis

A basis $v_1, \ldots, v_n$ for V determines a *dual basis* $v_1^*, \ldots, v_n^*$ for V^* where each $v_i^* \colon V \to \mathbb{R}$ is defined by

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Example. $V = \mathbb{R}^3$ with standard basis e_1, e_2, e_3. We have

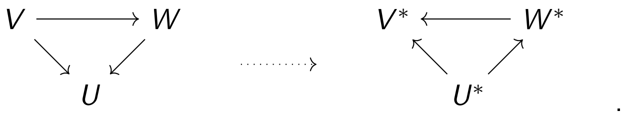
$$\begin{aligned} e_3^*(2,5,4) &= e_3^*(2e_1 + 5e_2 + 4e_3) \\ &= e_3^*(2e_1) + e_3^*(5e_2) + e_3^*(4e_3) \\ &= 2e_3^*(e_1) + 5e_3^*(e_2) + 4e_3^*(e_3) \\ &= 4. \end{aligned}$$

Dualization is a contravariant functor

Given a linear function $f\colon V \to W$, there is a natural induced function $f^*\colon W^* \to V^*$ define as follows: for $\phi \in W^*$, i.e., linear $\phi\colon W \to \mathbb{R}$, we let $f^*(\phi) = \phi \circ f$:

$$f^*(\phi) = \phi \circ f\colon V \xrightarrow{f} W \xrightarrow{\phi} \mathbb{R}.$$

Functoriality: (1) $\mathrm{id}_V\colon V \to V$ induces $\mathrm{id}_{V^*}\colon V^* \to V^*$, and (2) commutative diagrams are preserved:



Duality and tensors

- $(V^*)^{\otimes \ell} \simeq (V^{\otimes \ell})^*$
- $\operatorname{Sym}^\ell V^* \simeq (\operatorname{Sym}^\ell V)^*$
- $\Lambda^\ell V^* \simeq (\Lambda^\ell V)^*$

$(V^*)^{\otimes \ell} \simeq (V^{\otimes \ell})^*$

Every element of $(V^*)^{\otimes \ell}$ is a linear combination of elements of the form $\phi_1 \otimes \cdots \otimes \phi_\ell$ where $\phi_i \colon V \to \mathbb{R}$.

Send $\phi_1 \otimes \cdots \otimes \phi_\ell$ to the element of $(V^{\otimes \ell})^*$ defined by

$$v_1 \otimes \cdots \otimes v_\ell \mapsto \phi_1(v_1) \cdots \phi_\ell(v_\ell)$$

for all $v_1, \ldots, v_\ell \in V$.

$$\begin{aligned} (V^*)^{\otimes \ell} &\to (V^{\otimes \ell})^* \\ \phi_1 \otimes \cdots \otimes \phi_\ell &\mapsto \left[v_1 \otimes \cdots \otimes v_\ell \mapsto \textstyle\prod_{i=1}^{\ell} \phi_i(v_i)\right] \end{aligned}$$

$\mathrm{Sym}^{\ell} V^* \simeq (\mathrm{Sym}^{\ell} V)^*$

$$\mathrm{Sym}^{\ell} V^* \to (\mathrm{Sym}^{\ell} V)^*$$

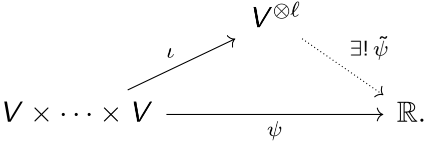
$$\phi_1 \cdots \phi_\ell \mapsto [v_1 \cdots v_\ell \mapsto \textstyle\sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{\ell} \phi_{\sigma(i)}(v_i)]$$

$\Lambda^\ell V^* \simeq (\Lambda^\ell V)^*$

$$\begin{aligned} \Lambda^\ell V^* &\to (\Lambda^\ell V)^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell &\mapsto [v_1 \wedge \cdots \wedge v_\ell \mapsto \det(\phi_i(v_j))] \end{aligned}$$

Forms

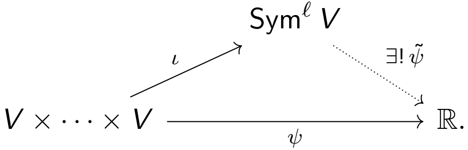
By an ℓ-form, we mean a multilinear function $\psi\colon \underbrace{V \times \cdots \times V}_{\ell \text{ times}} \to \mathbb{R}$. By the universal property of the tensor product,



So we can identify the ℓ-form ψ with $\tilde{\psi} \in (V^{\otimes \ell})^* \simeq (V^*)^{\otimes \ell}$.

Symmetric forms

A symmetric ℓ-form, is a symmetric multilinear function $\psi\colon \underbrace{V \times \cdots \times V}_{\ell \text{ times}} \to \mathbb{R}$. By the universal property of symmetric tensors,

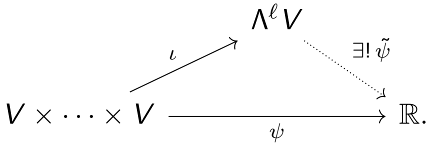


So we can identify the symmetric ℓ-form ψ with $\tilde{\psi} \in (\mathrm{Sym}^\ell V)^* \simeq \mathrm{Sym}^\ell V^*$.

Example. An inner product $\langle\ ,\ \rangle\colon V \times V \to \mathbb{R}$, can be identified with an element of $\mathrm{Sym}^2 V^*$. Challenge: The ordinary inner product on $\mathbb{R}^n$ corresponds to which element of $\mathrm{Sym}^2(\mathbb{R}^{n*})$?

Alternating forms

An alternating ℓ-form, is an alternating multilinear function $\psi\colon \underbrace{V \times \cdots \times V}_{\ell \text{ times}} \to \mathbb{R}$. By the universal property of alternating tensors,



So we can identify the alternating ℓ-form ψ with $\tilde{\psi} \in (\Lambda^\ell V)^* \simeq \Lambda^\ell V^*$.

Alternating forms

The vector space of alternating ℓ-forms on V is denoted $\mathrm{Alt}^{\ell} V$.

We have the identifications

$$\mathrm{Alt}^{\ell} V \simeq (\Lambda^{\ell} V)^* \simeq \Lambda^{\ell} V^*.$$

Pullbacks

Let $L\colon V \to W$ be linear, and let $L^*\colon W^* \to V^*$ be the pullback mapping. We get

$$L^*\colon (\Lambda^\ell W)^* \to (\Lambda^\ell V)^*$$
$$\eta \mapsto [v_1 \wedge \cdots v_\ell \mapsto \eta(Lv_1 \wedge \cdots Lv_\ell)]$$

$$\eta\colon \Lambda^\ell W \to \mathbb{R} \quad \text{and} \quad L^*\eta\colon \Lambda^\ell V \to \mathbb{R}$$

$$L^*\colon \Lambda^\ell W^* \to \Lambda^\ell V^*$$
$$\phi_1 \wedge \cdots \wedge \phi_\ell \mapsto L^*\phi_1 \wedge \cdots \wedge L^*\phi_\ell$$

$$L^*\colon \Lambda^\ell W^* \to (\Lambda^\ell V)^*$$
$$\phi_1 \wedge \cdots \wedge \phi_\ell \to [v_1 \wedge \cdots \wedge v_\ell \mapsto \det(\phi_i(Lv_j))]$$

Polynomials

There is an action of $\mathrm{Sym}^\ell V^*$ on V defined by $(\phi_1 \cdots \phi_\ell)(v) = \prod_{i=1}^{\ell} \phi_i(v)$.

Example. Let $V = \mathbb{R}^3$ with standard basis e_1, e_2, e_3. The dual basis is e_1^*, e_2^*, e_3^*. For $(x, y, z) \in \mathbb{R}^3$, we have, for example,

$$\begin{aligned} e_2^*(x, y, z) &= e_2^*(xe_1 + ye_2 + ze_3) \\ &= xe_2^*(e_1) + ye_2^*(e_2) + ze_2^*(e_3) \\ &= y. \end{aligned}$$

In general, e_i^* is the i-th projection function. Using the action, above, we have, for example,

$$((e_2^*)^2 e_3^*)(x, y, z) = e_2^*(x, y, z) e_2^*(x, y, z) e_3^*(x, y, z) = y^2 z.$$

For instance, $((e_2^*)^2 e_3^*)(7, 4, 1) = 4 \cdot 4 \cdot 1 = 16$.

Polynomials

Action of $\mathrm{Sym}^\ell V^*$ on V: $\quad (\phi_1 \cdots \phi_\ell)(v) = \prod_{i=1}^\ell \phi_i(v)$. We can extend this action to an action of $\mathrm{Sym}\, V^* = \bigoplus_{\ell \geq 0} \mathrm{Sym}\, V^*$ on V.

For example, consider the element $f = 2(e_1^*)^2 + 5e_1^*(e_2^*)^2(e_3^*) \in \mathrm{Sym}(\mathbb{R}^3)^*$. We have

$$f(x, y, z) = 2x^2 + 5xy^2z.$$

Point: $\mathrm{Sym}\, V^*$ is a coordinate-free way to think of polynomials acting on V. Choosing an ordered basis for V identifies V with $\mathbb{k}^n$ and $\mathrm{Sym}\, V^*$ with $\mathbb{k}[x_1, \ldots, x_n]$.

Exercises

Consider

$$L \colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto (y, x, 2x + y)$$

Take standard bases e_1, e_2 for $\mathbb{R}^2$ and f_1, f_2, f_3 for $\mathbb{R}^3$. Taking dual bases allows us to identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ and $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$.

What is the matrix representing $L^* \colon \mathbb{R}^3 \to \mathbb{R}^2$?

Exercises

$$L\colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto (y, x, 2x + y)$$

There is an induced mapping $L^*\colon \Lambda^2(\mathbb{R}^3)^* \to \Lambda^2(\mathbb{R}^2)^*$.

Ordered basis for $\Lambda^2(\mathbb{R}^3)^*$: $f_1^* \wedge f_2^*,\ f_1^* \wedge f_3^*,\ f_2^* \wedge f_3^*$.

Ordered basis for $\Lambda^2(\mathbb{R}^2)^*$: $e_1^* \wedge e_2^*$.

Choosing these bases, $L^*\colon \Lambda^2(\mathbb{R}^3)^* \to \Lambda^2(\mathbb{R}^2)^*$ becomes a mapping $L^*\colon \mathbb{R}^3 \to \mathbb{R}$. What is the matrix representing this mapping?

Exercises

$$L\colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto (y, x, 2x + y)$$

Induced mapping $L^*\colon \Lambda^2(\mathbb{R}^3)^* \to \Lambda^2(\mathbb{R}^2)^*$.

Images of standard basis vectors for $\Lambda^2(\mathbb{R}^3)^*$:

$$f_1^* \wedge f_2^* \mapsto L^* f_1^* \wedge L^* f_2^* = (f_1^* \circ L) \wedge (f_2^* \circ L) = e_2^* \wedge e_1^* = -e_1^* \wedge e_2^*$$

$$f_1^* \wedge f_3^* \mapsto e_2^* \wedge (2e_1^* + e_2^*) = -2e_1^* \wedge e_2^*$$

$$f_2^* \wedge f_3^* \mapsto e_1^* \wedge (2e_1^* + e_2^*) = e_1^* \wedge e_2^*.$$

Matrix: $\begin{bmatrix} -1 & -2 & 1 \end{bmatrix}$.

Exercises

Thus, we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{R}^3)^* & \xrightarrow{L^*} & \mathbb{R}^* \\ \wr\downarrow & & \wr\downarrow \\ \mathbb{R}^3 & \xrightarrow{A} & \mathbb{R} \end{array}$$

where

$$A = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}.$$