Tensors

Preliminaries

- Remember the quiz on Wednesday.
- Comments on HW.
  - Use `\colon` instead of : when defining functions:

$$f\colon U \to V$$
$$f : U \to V$$

  - Proofs consist of complete sentences.

Left over from last time

Standard basis for T_pM with respect to a chart (U, h):

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

$T_p^{geo}(M): \quad (\frac{\partial}{\partial x_i})_p = [t \mapsto h^{-1}(h(p) + te_i)]$

$T_p^{alg}: \quad (\frac{\partial}{\partial x_i})_p \, f \mapsto \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p))$

$T_p^{phy}: \quad (\frac{\partial}{\partial x_i})_p \, (U, h) \mapsto e_i$

Tensors

Three types:

- ordinary $v \otimes w$
- symmetric $v \cdot w$
- alternating $v \wedge w$

Example

U, V, W finite dimensional $\mathbb{k}$-vector spaces

$$U \otimes V \otimes W = \mathrm{Span}_{\mathbb{k}}\{u \otimes v \otimes w : u \in U, v \in V, w \in W\},$$

where $\otimes$ is "multilinear", e.g.,

$$\begin{aligned} u \otimes (2v + v') \otimes w &= u \otimes (2v) \otimes w + u \otimes v' \otimes w \\ &= 2\, u \otimes v \otimes w + u \otimes v' \otimes w. \end{aligned}$$

Another example of multilinearity:

$$5(u \otimes v \otimes w) = (5u) \otimes v \otimes w = u \otimes (5v) \otimes w = u \otimes v \otimes (5w).$$

Construction of $V \otimes W$

$F(V, W) =$ vector space with basis the symbols $[v, w]$ where $v \in V$ and $w \in W$.

No relations among symbols! For example,

$$[v + v', w] \neq [v, w] + [v', w].$$

Construction of $V \otimes W$

Let T be the subspace of $F(V, W)$ spanned by:

$$
\begin{aligned}
&[v + v', w] - [v, w] - [v', w] \\
&[v, w + w'] - [v, w] - [v, w'] \\
&[\lambda v, w] - \lambda[v, w] \\
&[v, \lambda w] - \lambda[v, w]
\end{aligned}
$$

for all $v, v \in V$, $w, w' \in W$, and $\lambda \in \mathbb{k}$.

Definition: $V \otimes W = F(V, W)/T$, and $v \otimes w = [v, w] \bmod T$.

Bases

Say we have bases:

$$V\colon\ e_1, \ldots, e_m$$
$$W\colon\ f_1, \ldots, f_n$$

Then $V \otimes W$ has basis $e_i \otimes f_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Therefore, $\dim V \otimes W = \dim V \dim W$.

The same holds for tensor products of more than two vector spaces, e.g,

$$\dim U \otimes V \otimes W = \dim U \dim V \dim W.$$

Example

Consider $\mathbb{R}^2$ with standard basis e_1, e_2. Then $\mathbb{R}^2 \otimes \mathbb{R}^2$ has basis:
$e_1 \otimes e_1, \quad e_1 \otimes e_2, \quad e_2 \otimes e_1, \quad e_2 \otimes e_2.$

Exercise: Express $(2,3) \otimes (1,4)$ in terms of this basis.

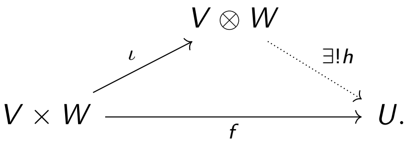
True or False: $(2,4) \otimes (1,3) = (1,2) \otimes (2,6)$.

Universal property of the tensor product

There is a *bilinear* mapping

$$\iota\colon V \times W \to V \otimes W$$
$$(v, w) \mapsto v \otimes w.$$

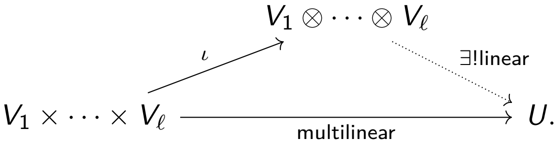
The tensor product $V \otimes W$ is characterized (up to isomorphism of vector spaces) by the following universal property: Given any bilinear mapping $f\colon V \times W \to U$ to a vector space U, there exists a unique linear mapping $h\colon V \otimes W \to U$ such that the following diagram commutes:



Thus, the tensor product turns "bilinear mappings into linear mappings".

Universal property of the tensor product

More generally, there is a similar commutative diagram that relates a multilinear mapping $V_1 \times \cdots \times V_n \to U$ with a linear mapping $V_1 \otimes \cdots \otimes V_n \to U$:



Symmetric tensors

$$V^{\otimes \ell} := \underbrace{V \otimes \cdots \otimes V}_{\ell \text{ times}}$$

For $1 \leq i \neq j \leq \ell$, define

$$t_{ij} = v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_\ell \\ - v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_\ell$$

and let $T' = \mathrm{Span}\{t_{ij} : 1 \leq i \neq j \leq \ell\}$.

Then $\mathrm{Sym}^\ell V := V^{\otimes \ell}/T'$, and
$v_1 \cdots v_\ell := [v_1 \otimes \cdots \otimes v_\ell] \in \mathrm{Sym}^\ell V$.

Exercise: In $\mathrm{Sym}^2 \mathbb{R}^2$, compute $(2,3) \cdot (1,4)$ in terms of the standard basis e_1, e_2 for $\mathbb{R}^2$.

Symmetric tensors

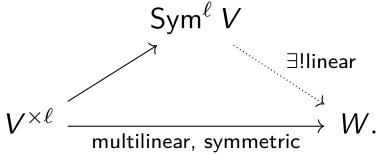
If V has basis $e_1, \ldots, e_n$, then $\mathrm{Sym}^\ell V$ has basis

$$\{e_1^{\alpha_1} \cdots e_n^{\alpha_n} : \alpha_i \geq 0 \text{ for all } i, \text{ and } \alpha_1 + \cdots + \alpha_n = \ell\}.$$

We have $\dim \mathrm{Sym}^\ell V = \binom{n+\ell-1}{\ell}$.

Universal property of symmetric products

Given a multilinear symmetric mapping $V^{\times \ell} \to W$ from the ℓ-fold Cartesian product of V with itself to W, there exists a unique linear mapping $\operatorname{Sym}^\ell V \to W$ making the following diagram commute:



The mapping $V^{\times \ell} \to \operatorname{Sym}^\ell V$ is the natural one determined by $(v_1, \ldots, v_\ell) \mapsto v_1 \cdots v_\ell$.

Alternating tensors

$\Lambda^\ell V = V^{\otimes \ell}/T''$ where

$$T'' = \mathrm{Span}\{v_1 \otimes \cdots \otimes v_\ell : v_i = v_j \text{ for some } i \neq j\}.$$

Notation: $v_1 \wedge \cdots \wedge v_\ell := [v_1 \otimes \cdots \otimes v_\ell]$.

Exercise. Show

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_\ell = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_\ell$$

Idea of proof: We know $0 = u \wedge (v + w) \wedge (v + w)$. Expand the RHS.

Exercise. In $\Lambda^2\mathbb{R}^2$ expand $(2, 3) \wedge (1, 4)$ in terms of the standard basis e_1, e_2 for $\mathbb{R}^2$.

Alternating tensors

If V has basis $e_1, \ldots, e_n$, then $\Lambda^\ell V$ has basis consisting of the vectors

$$e_\mu := e_{\mu_1} \wedge \cdots \wedge e_{\mu_\ell} \text{ such that } \mu_1 < \cdots < \mu_\ell.$$

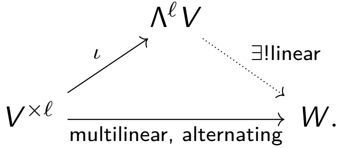
Therefore, $\dim \Lambda^\ell V = \binom{n}{\ell}$.

Universal property of alternating products

A multilinear map $f\colon V^{\times \ell} \to W$ is *alternating* if $f(v_1, \ldots, v_\ell) = 0$ if $v_i = v_j$ for some $i \neq j$. There is a canonical multilinear alternating mapping

$$\begin{aligned} \iota\colon V^{\times \ell} &\to \Lambda^\ell V \\ (v_1, \ldots, v_\ell) &\mapsto v_1 \wedge \cdots \wedge v_\ell. \end{aligned}$$

The ℓ-th exterior product is characterized by the following universal property: given any vector space W and multilinear alternating mapping $V^{\times \ell} \to W$, there is a unique linear map $\Lambda^\ell V \to W$ making the following diagram commute:



Algebra structure on tensors

$$V^{\otimes r} \otimes V^{\otimes s} \to V^{\otimes(r+s)}$$
$$(u_1 \otimes \cdots \otimes u_r) \otimes (v_1 \otimes \cdots \otimes v_s) \mapsto u_1 \otimes \cdots \otimes u_r \otimes v_1 \otimes \cdots \otimes v_s$$

Similarly, we have

$$\operatorname{Sym}^r V \times \operatorname{Sym}^s V \to \operatorname{Sym}^{r+s} V$$
$$\Lambda^r V \times \Lambda^s V \to \Lambda^{r+s} V.$$

Finally, we define:

$$\operatorname{Sym} V := \bigoplus_{\ell \geq 0} \operatorname{Sym}^\ell V \text{ and } \quad \Lambda^\bullet V := \bigoplus_{\ell \geq 0} \Lambda^\ell V.$$