Tangent space

Quiz

1. Let X be a topological space. What is an *n-dimensional chart* on X?
2. Given charts (U, h) and (V, k), what is the *transition function* from U to V? Make sure to specify the domain and codomain.
3. What does it mean for charts (U, h) and (V, k) at a point p to be *differentiably related*?
4. What does it mean for a continuous function $f : M \to N$ between manifolds to be *differentiable at* $p \in M$?

Three versions of tangent space

How do we define the tangent space for a manifold that is not embedded in space?

In the spirit of manifolds, tangent space should be an abstract vector space, without inherent coordinates.

Three takes:

$T_pM = T_p^{\mathrm{geo}}M$: tangents to curves

$T_pM = T_p^{\mathrm{alg}}M$: directional derivatives

$T_pM = T_p^{\mathrm{phy}}M$: vectors in $\mathbb{R}^n$ (use charts), compatible via transition functions

$T_p^{\mathrm{geo}} M$

$\mathcal{K}_p(M) := \{\alpha\colon (-\varepsilon, \varepsilon) \to M \mid$
α is differentiable, $\varepsilon > 0$, and $\alpha(0) = p\}$.

For $\alpha, \beta \in \mathcal{K}_p(M)$ say $\alpha \sim \beta$, if

$$(h \circ \alpha)'(0) = (h \circ \beta)'(0) \in \mathbb{R}^n$$

for some (hence any) chart (U, h) around p.

Definition:

$$T_p^{\mathrm{geo}} M := \mathcal{K}_p(M)/\sim .$$

Fixing a chart (U, h) gives a bijection

$$\phi\colon\ T_p^{\mathrm{geo}} \to \mathbb{R}^n$$
$$[\alpha] \mapsto (h \circ \alpha)'(0).$$

Use ϕ to impose a linear structure on $T_p^{\mathrm{geo}} M$.

T^{alg}

If f, g are two differentiable $\mathbb{R}$-valued functions defined in a neighborhood of $p \in M$, say $f \sim g$ if $f = g$ when restricted to *some* neighborhood of p.

An equivalence class, $[f]$, is call a *germ* of a differentiable function at p. Let $\mathcal{E}_p$ denote the germs of functions at p.

Definition. An *(algebraically-defined) tangent vector* to M at p, is a *derivation* of the ring $\mathcal{E}_p(M)$ of germs, that is, a linear map on the germs

$$v \colon \mathcal{E}_p(M) \to \mathbb{R}$$

satisfying the product rule

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g)$$

for all $f, g \in \mathcal{E}_p(M)$. Then $T_p^{\mathrm{alg}}(M)$ is the vector space of these derivations.

T_p^{phy}

Charts at p on manifold with differentiable structure $\mathcal{D}$:

$$\mathcal{D}_p(M) := \{(U, h) \in \mathcal{D} \mid p \in U\}.$$

Definition. $T_p^{\mathrm{phy}} M$ is the collection of mappings

$$\begin{aligned} v\colon \mathcal{D}_p &\to \mathbb{R}^n \\ (U, h) &\mapsto v(U, h) \end{aligned}$$

such that for every pair of charts (U, h) and (V, k),

$$v(V, k) = D_{h(p)}(k \circ h^{-1})(v(U, h)).$$

Fix a chart (U, h) to get a bijection $\phi\colon T_p^{\mathrm{phy}} \to \mathbb{R}^n$ and impose a linear structure.