



Tangent space

Preliminaries

- ▶ This week's reading.
- ▶ Wednesday's quiz.
- ▶ Last week's HW.

Three versions of tangent space

How do we define the tangent space for a manifold that is not embedded in space?

In the spirit of manifolds, tangent space should be an abstract vector space, without inherent coordinates.

Three takes:

$T_p M = T_p^{\text{geo}} M$: tangents to curves

$T_p M = T_p^{\text{alg}} M$: directional derivatives

$T_p M = T_p^{\text{phy}} M$: vectors in $\mathbb{R}^n$ (use charts), compatible via transition functions

$$T_p^{\text{geo}} M$$

$\mathcal{K}_p(M) := \{ \alpha : (-\varepsilon, \varepsilon) \rightarrow M \mid$
 $\alpha \text{ is differentiable, } \varepsilon > 0, \text{ and } \alpha(0) = p \}.$

For $\alpha, \beta \in \mathcal{K}_p(M)$ say $\alpha \sim \beta$, if

$$(h \circ \alpha)'(0) = (h \circ \beta)'(0) \in \mathbb{R}^n$$

for some (hence any) chart (U, h) around p .

Definition:

$$T_p^{\text{geo}} M := \mathcal{K}_p(M) / \sim .$$

Fixing a chart (U, h) gives a bijection

$$\begin{aligned} \phi : T_p^{\text{geo}} &\rightarrow \mathbb{R}^n \\ [\alpha] &\mapsto (h \circ \alpha)'(0). \end{aligned}$$

Use ϕ to impose a linear structure on $T_p^{\text{geo}} M$.

If f, g are two differentiable $\mathbb{R}$ -valued functions defined in a neighborhood of $p \in M$, say $f \sim g$ if $f = g$ when restricted to *some* neighborhood of p .

An equivalence class, $[f]$, is call a *germ* of a differentiable function at p . Let $\mathcal{E}_p$ denote the germs of functions at p .

Definition. An (*algebraically-defined*) *tangent vector* to M at p , is a *derivation* of the ring $\mathcal{E}_p(M)$ of germs, that is, a linear map on the germs

$$v: \mathcal{E}_p(M) \rightarrow \mathbb{R}$$

satisfying the product rule

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g)$$

for all $f, g \in \mathcal{E}_p(M)$. Then $T_p^{\text{alg}}(M)$ is the vector space of these derivations.

T_p^{phy}

Charts at p on manifold with differentiable structure $\mathcal{D}$:

$$\mathcal{D}_p(M) := \{(U, h) \in \mathcal{D} \mid p \in U\}.$$

Definition. $T_p^{\text{phy}} M$ is the collection of mappings

$$\begin{aligned} v: \mathcal{D}_p &\rightarrow \mathbb{R}^n \\ (U, h) &\mapsto v(U, h) \end{aligned}$$

such that for every pair of charts (U, h) and (V, k) ,

$$v(V, k) = D_{h(p)}(k \circ h^{-1})(v(U, h)).$$

Fix a chart (U, h) to get a bijection $\phi: T_p^{\text{phy}} \rightarrow \mathbb{R}^n$ and impose a linear structure.