Tangent space, tangent mappings

Three versions of tangent space

Geometric (curves): $[\alpha] \in T_p^{\mathrm{geo}}$, $\alpha \sim \beta$ if $(h \circ \alpha)'(0) = (h \circ \beta)'(0)$

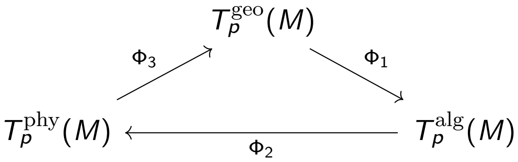
Algebraic (derivations): $v \in T_p^{\mathrm{alg}}$, $v \colon \mathcal{E}_p \to \mathbb{R}$, linear + product rule

Physical (compatible vectors): $v \in T_p^{\mathrm{phy}}$, $v \colon \text{Charts} \to \mathbb{R}^n$

$$v(V, k) = D_p(k \circ h^{-1})(v(U, h))$$

Equivalence of different notions of tangent space

Let M be an n-dimensional manifold and let $p \in M$. There are canonical (i.e., do not involve choosing bases) linear isomorphisms Φ_1, Φ_2, Φ_3 that make the following diagram commute.



Equivalence of different notions of tangent space

An example of a detail to be checked: Is Φ_1 well-defined? Suppose $\alpha \sim \beta$. Let f be a germ of a function at p. Question: Is $(f \circ \alpha)'(0) = (f \circ \beta)'(0)$?

Yes. Sketch of proof: Choose chart (U, h) at p. We are given that $(h \circ \alpha)'(0) = (h \circ \beta)'(0)$. Apply the chain rule:

$$
\begin{aligned}
(f \circ \alpha)'(0) &= ((f \circ h^{-1}) \circ (h \circ \alpha))'(0) \\
&= J(f \circ h^{-1})(h(p))J(h \circ \alpha))(0) \\
&= J(f \circ h^{-1})(h(p))J(h \circ \beta))(0) \\
&= J((f \circ h^{-1}) \circ (h \circ \beta))(0) \\
&= J(f \circ \beta)(0) \\
&= (f \circ \beta)'(0).
\end{aligned}
$$

Tangent mappings

Given a (differentiable) mapping $f\colon M \to N$ between manifolds, and $p \in M$, there exists a linear mapping

$$df_p\colon T_pM \to T_{f(p)}N.$$

Tangent mappings, geometric

Let $[\alpha] \in T_p^{\mathrm{geo}}(M)$. How can we use $f\colon M \to N$ to define an element of $T_{f(p)}^{\mathrm{geo}}(N)$?

Answer:

$$\begin{aligned} df_p\colon T_p^{\mathrm{geo}} M &\to T_{f(p)}^{\mathrm{geo}} N \\ [\alpha] &\mapsto [f \circ \alpha]. \end{aligned}$$

Tangent mappings, algebraic

Precomposing by $f\colon M \to N$ defines an algebra homomorphism called the pullback along f:

$$\begin{aligned} f^*\colon \mathcal{E}_{f(p)}(N) &\to \mathcal{E}_p(M) \\ [\phi] &\mapsto [\phi \circ f]. \end{aligned}$$

Algebraic version of the tangent mapping:

$$\begin{aligned} df_p\colon T_p^{\mathrm{alg}} M &\to T_{f(p)}^{\mathrm{alg}} N \\ v &\mapsto v \circ f^*. \end{aligned}$$

Tangent mappings, physical

Let $v \in T_p^{\mathrm{phy}}$. We need to define $df_p(v) \in T_{f(p)}^{\mathrm{phy}}$. Given a chart (V, k) at $f(p)$, choose a chart (U, h) at p such that $f(U) \subseteq V$. Then define

$$(df_p(v))(V, k) := D_{h(p)}(k \circ f \circ h^{-1})v(U, h).$$

This gives the desired mapping

$$df_p \colon T_p^{\mathrm{phy}} \to T_{f(p)}^{\mathrm{phy}}.$$

Standard basis for $T_p(M)$ with respect to a chart (U, h)

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

Geometric (curves version)

$\mathcal{K}_p(M) := \{\alpha\colon (-\varepsilon, \varepsilon) \to M \mid$
α is differentiable, $\varepsilon > 0$, and $\alpha(0) = p\}$.

$\alpha \sim \beta$ if $(h \circ \alpha)'(0) = (h \circ \beta)'(0) \in \mathbb{R}^n$ for some (hence any) chart (U, h) around p.

Then $T_p^{\text{geo}} M := \mathcal{K}_p(M)/\sim$.

Definition. $\left(\frac{\partial}{\partial x_i}\right)_p$ is the equivalence class of
$\alpha(t) = h^{-1}(h(p) + te_i)$
(t in a nbd. of 0, and $e_i \in \mathbb{R}^n$ is the i-th std. basis vector)

Standard basis for $T_p(M)$ with respect to a chart (U, h)

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

Algebraic (derivation version)

$\mathcal{E}_p =$ germs of differentiable functions at p.

An *(algebraically-defined) tangent vector* to M at p, is a *derivation* of the ring $\mathcal{E}_p(M)$ of germs, that is, a linear map on the germs $v\colon \mathcal{E}_p(M) \to \mathbb{R}$ satisfying the product rule

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g)$$

for all $f, g \in \mathcal{E}_p(M)$. Then $T_p^{\mathrm{alg}}(M)$ is the vector space of these derivations.

Definition. $\left(\frac{\partial}{\partial x_i}\right)_p$ is the derivation defined by $\left(\frac{\partial}{\partial x_1}\right)_p f := \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p))$.

Standard basis for $T_p(M)$ with respect to a chart (U, h)

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

Physical

$T_p^{\mathrm{phy}} M$ is the collection of mappings

$$v \colon \mathcal{D}_p \to \mathbb{R}^n$$
$$(U, h) \mapsto v(U, h)$$

such that for every pair of charts (U, h) and (V, k),

$$v(V, k) = D_{h(p)}(k \circ h^{-1})(v(U, h)).$$

Definition. $\left(\frac{\partial}{\partial x_i}\right)_p$ is the (physical) tangent vector such that $(U, h) \mapsto e_i$.