Definition of a manifold

Definition

An *n-dimensional differentiable manifold* is a pair $(M, \mathcal{D})$ consisting of a second countable Hausdorff topological space M with an n-dimensional differentiable structure $\mathcal{D}$.

Essential vocabulary:

- charts (U, h), locally Euclidean, choosing coordinates
- transition function, change of coordinates
- differentiably related charts (U, h) and (V, k)
- differentiable atlas, equivalent atlases
- differentiable structure

Projective plane

$$\begin{aligned}\mathbb{P}^2 &= \text{ one-dimensional subspaces of } \mathbb{R}^3 \\ &= \text{ lines through the origin in } \mathbb{R}^3 \\ &= \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ [(x,y,z) \sim \lambda(x,y,z) \text{ for } \lambda \neq 0] \,.\end{aligned}$$

quotient topology

We say (x, y, z) are the *homogeneous coordinates* for a point in $\mathbb{P}^2$.

Quiz question 1 (out of 2)

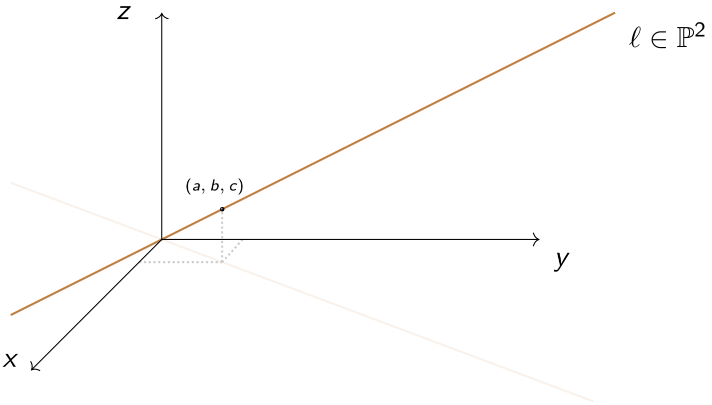
Which of the following represent the point $(2, 4, 5) \in \mathbb{P}^2$?

(a) $(4, 8, 10)$
(b) $(1, 2, 2)$
(c) $(1/2, 1, 5/4)$.

Solution: (a) and (c).

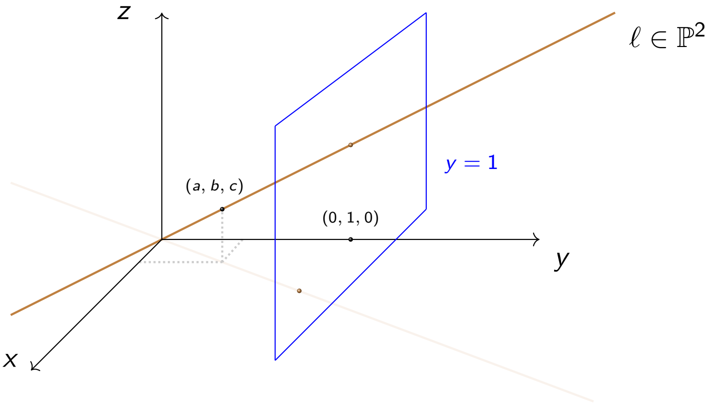
Projective plane: standard charts

$$\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ [(x,y,z) \sim \lambda(x,y,z) \text{ for } \lambda \neq 0]$$

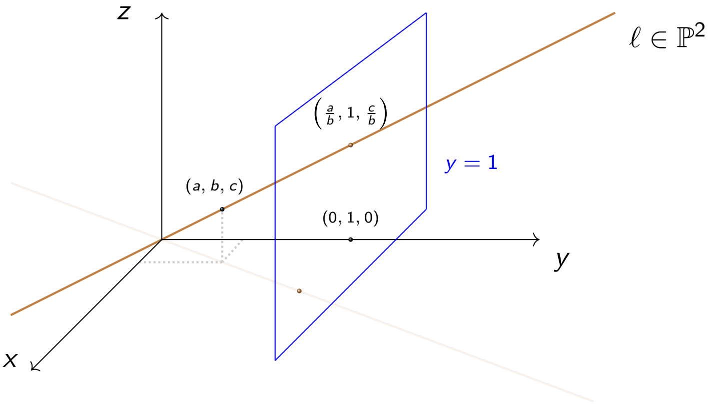


Projective plane: standard charts

$$\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ [(x,y,z) \sim \lambda(x,y,z) \text{ for } \lambda \neq 0]$$



Projective plane: standard charts



$$(a, b, c) = \left(\frac{a}{b}, 1, \frac{c}{b}\right) \mapsto \left(\frac{a}{b}, \frac{c}{b}\right)$$

Projective plane: standard charts

$$\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ [(x,y,z) \sim \lambda(x,y,z) \text{ for } \lambda \neq 0]$$

$$U_y := \left\{(x,y,z) \in \mathbb{P}^2 : y \neq 0\right\}$$

$$\phi_y \colon U_y \to \mathbb{R}^2$$
$$(a,b,c) \mapsto \left(\frac{a}{b}, \frac{c}{b}\right)$$

$$\phi_y^{-1}(u,v) = (u,1,v)$$

Projective plane: standard charts

$$\mathbb{P}^2 = \mathbb{R}^3 \setminus \{(0,0,0)\} \Big/ [(x,y,z) \sim \lambda(x,y,z) \text{ for } \lambda \neq 0]$$

$$U_x, \qquad U_y, \qquad U_z$$

$$\phi_x \colon U_x \to \mathbb{R}^2 \qquad \phi_y \colon U_y \to \mathbb{R}^2 \qquad \phi_z \colon U_z \to \mathbb{R}^2$$

$$(a,b,c) \mapsto \left(\frac{b}{a}, \frac{c}{a}\right) \qquad (a,b,c) \mapsto \left(\frac{a}{b}, \frac{c}{b}\right) \qquad (a,b,c) \mapsto \left(\frac{a}{c}, \frac{b}{c}\right)$$

$$\phi_x^{-1}(u,v) = (1,u,v) \qquad \phi_y^{-1}(u,v) = (u,1,v) \qquad \phi_z^{-1}(u,v) = (u,v,1)$$

standard atlas: $\mathcal{U} = \{(U_x, \phi_x), (U_y, \phi_y), (U_z, \phi_z)\}$

Quiz question 2 (out of 2)

What are the coordinates of $(4, 2, 16) \in \mathbb{P}^2$ with respect to the chart (U_y, ϕ_y):

(a) $(2, 8)$

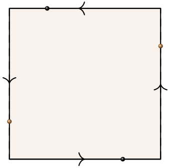
(b) $(2, 1)$

(c) $(1/2, 4)$.

Solution: (a).

Other models for $\mathbb{P}^2$

- 2-sphere w/ antipodal points identified: $\mathbb{P}^2 = S^2/[p \sim -p]$
- upper hemisphere w/antipodal points on equator identified
- closed square with "antipodal" boundary points identified



Projective n-space

$$\begin{aligned}
\mathbb{P}^n &= \text{ one-dimensional subspaces of } \mathbb{R}^{n+1} \\
&= \text{ lines through the origin in } \mathbb{R}^{n+1} \\
&= \mathbb{R}^{n+1} \setminus \{0\} \Big/ [p \sim \lambda p \text{ for } \lambda \neq 0] \\
&= S^n/[p \sim -p]
\end{aligned}$$

Projective n-space

Charts

$U_i = \{(x_1, \ldots, x_{n+1}) : x_i \neq 0\}$ for $i = 1, \ldots, n+1$

$$\begin{aligned} \phi_i \colon U_i &\to \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) &\mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right) \end{aligned}$$

Thanks!