Differentiable Mappings

The discussion of differentiable mappings of manifolds will take place on the blackboard.

These slides will cover a few side topics:

1. Review of projective n-space, $\mathbb{P}^n$.
2. Open functions.
3. Compactness.

Projective n-space

$$\begin{aligned} \mathbb{P}^n &= \text{ one-dimensional subspaces of } \mathbb{R}^{n+1} \\ &= \text{ lines through the origin in } \mathbb{R}^{n+1} \\ &= \mathbb{R}^{n+1} \setminus \{0\} \Big/ [p \sim \lambda p \text{ for } \lambda \neq 0] \\ &= S^n/[p \sim -p] \end{aligned}$$

Projective n-space

Charts

$U_i = \{(x_1, \dots, x_{n+1}) : x_i \neq 0\}$ for $i = 1, \dots, n+1$

$$\begin{aligned} \phi_i \colon U_i &\to \mathbb{R}^n \\ (x_1, \dots, x_{n+1}) &\mapsto \left(\frac{x_1}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right) \end{aligned}$$

Open mappings

Definition. Let $f: S \to T$ be a function between topological spaces.

- f is *continuous* if $f^{-1}(V)$ is open for all open subsets $V \subseteq T$.
- f is *open* if $f(U)$ is open for all open subsets $U \subseteq S$.

What about following function?

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Point: Not all topological properties are preserved under continuous mappings.

Compactness

Definition. A topological space S is *compact* if every open covering of S has a finite subcover:

If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open subsets of S such that $S = \cup_{\alpha \in I} U_\alpha$, then there exists a finite subset $\{i_1, \ldots, i_k\} \subseteq I$ such that $S = U_{i_1} \cup \cdots \cup U_{i_k}$.

Examples. Is the open interval $(0,1) \subset \mathbb{R}$ compact? What about the closed interval $[0,1] \subset \mathbb{R}$? By Math 321, a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Compactness

Proposition. Compactness is preserved by continuous mappings: Suppose $f\colon S \to T$ is a continuous mapping topological spaces and C is a compact subset of S. The $f(C) \subseteq T$ is compact.

Proof. Exercise.

Thanks!