Differential multivariable calculus

$U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$

$f : U \to V$ is differentiable at $p \in U$ if there exists a linear function

$$Df_p : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{h \to 0} \frac{|f(p+h) - f(p) - Df_p(h)|}{|h|} = 0.$$

(How does this compare with the definition from 1-variable calculus?)

For $|h|$ small,

$$|f(p+h) - f(p) - Df_p(h)| \approx 0$$

So

$$f(p+h) \approx f(p) + Df_p(h).$$

If $h \approx 0$, then $f(p) + Df_p(h)$ approximates f near p.

Best affine approximation of f at p:

$$Af_p(x) = f(p) + Df_p(x)$$

or

$$Af_p(x) = f(p) + Df_p(x - p).$$

In the latter, if $x \approx p$, then $f(p) + Df_p(x - p)$ approximates f near p.

The derivative Df_p has matrix

$$Jf_p := \begin{bmatrix} \frac{\partial f_1(p)}{\partial x_1} & \cdots & \frac{\partial f_1(p)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(p)}{\partial x_1} & \cdots & \frac{\partial f_m(p)}{\partial x_n} \end{bmatrix}.$$

Interpretations of the derivative

$f : \mathbb{R}^k \to \mathbb{R}^n$, $k < n$: parametrized k-surface in $\mathbb{R}^n$

columns of $Jf(p)$ are tangent vectors at $f(p)$

$f : \mathbb{R}^n \to \mathbb{R}$: (scalar-valued) function on $\mathbb{R}^n$

$Jf(p)$ has a single row: $\mathrm{grad}\, f(p) = \nabla f(p)$.

$f : \mathbb{R}^n \to \mathbb{R}^n$: vector field on $\mathbb{R}^n$

$Jf(p)$ gives a linearization of the vector field

Interpretations of the derivative

$\boxed{k < n}$

$$f \colon \mathbb{R}^k \to \mathbb{R}^n$$

Then f is a parametrization of a k-dimensional surface in $\mathbb{R}^n$.

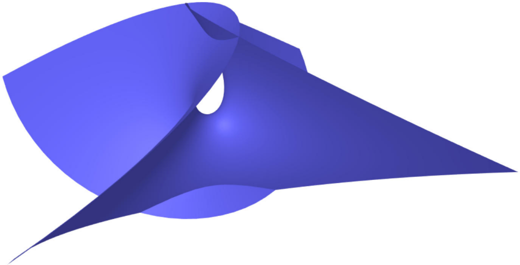
$k = 1$ – parametrized curve

$k = 2$ – parametrized surface.

The columns of $Jf(p)$ span the tangent space at p (if $Jf(p)$ has full rank).

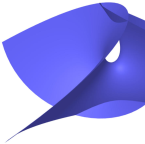
Example of parametrized surface

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto (x - x^3/3 + xy^2, y - y^3/3 + yx^2, x^2 - y^2)$$



Example of parametrized surface

$$f(x,y) = (x - x^3/3 + xy^2, y - y^3/3 + yx^2, x^2 - y^2)$$



$$Jf = \begin{pmatrix} 1 - x^2 + y^2 & 2\,xy \\ 2\,xy & 1 + x^2 - y^2 \\ 2\,x & -2\,y \end{pmatrix}, \qquad Jf(0, 0.8) = \begin{pmatrix} 1.64 & 0 \\ 0 & 0.36 \\ 0 & -1.6 \end{pmatrix}$$

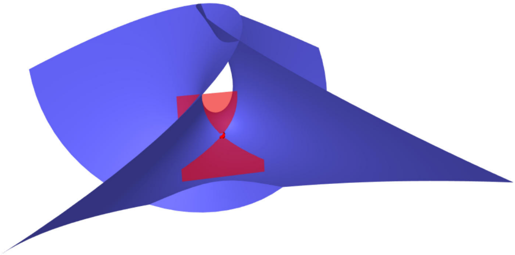
$$Af_p(x,y) = f(p) + Jf(p)\begin{pmatrix} x \\ y \end{pmatrix} = (0, -0.629\bar{3}, -0.64) + x(1.64, 0, 0) +$$

$$= (1.64x, -0.629\bar{3} + 0.36y, -0.64 - 1.6y)$$

Example of parametrized surface

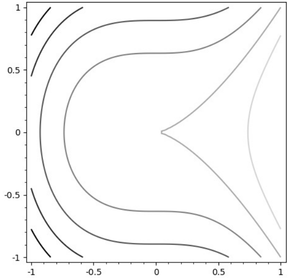
$$f(x,y) = (x - x^3/3 + xy^2, y - y^3/3 + yx^2, x^2 - y^2)$$

$$Af_p(x,y) = (1.64x, -0.629\overline{3} + 0.36y, -0.64 - 1.6y)$$

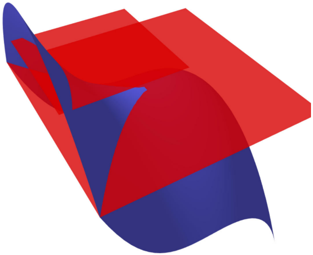


Example of scalar-valued function

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto x^3 - y^2$$



contour plot of f



level sets of $g(x, y) = (x, y, f(x, y))$

Quiz question 1 (of 2)

Which of the following is a tangent vector to the surface parametrized by

$$f(x, y) = (x + y, x^2 + y^2, xy)$$

at the point $p = (1, 2)$?

1. $(1, 4, 1)$
2. $(1, 2, 3)$
3. $(1, 4, 2)$.

Quiz question 1 (of 2)

$$f(x, y) = (x + y, x^2 + y^2, xy), \qquad p = (1, 2).$$

1. $(1, 4, 1)$
2. $(1, 2, 3)$
3. $(1, 4, 2)$.

$$Jf = \begin{pmatrix} 1 & 1 \\ 2x & 2y \\ y & x \end{pmatrix}, \qquad Jf(1, 2) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 2 & 1 \end{pmatrix},$$

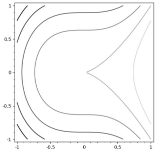
Example of scalar-valued function

Recall the main properties of the gradient $\nabla f(p)$:

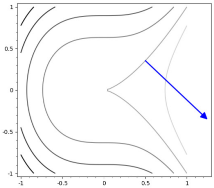
- It's perpendicular to the level set containing $f(p)$.
- It points in the direction of quickest increase of f at p.
- It's length is rate of growth of f in that direction.

Example of scalar-valued function

$$f(x, y) = x^3 - y^2$$



$$\nabla f = (3x^2, -2y), \qquad \nabla f(0.5, (0.5)^{3/2}) \approx (0.5, 0.707)$$



Quiz question 2 (of 2)

What is the rate of quickest increase of the function

$$f(x, y, z) = x^3 - xy + z^2 + 2$$

at the point $(1, 2, 3)$?

1. $\sqrt{15}$
2. 4
3. $\sqrt{38}$.

Quiz question 2 (of 2)

$$f(x, y, z) = x^3 - xy + z^2 + 2, \quad p = (1, 2, 3)$$

1. $\sqrt{15}$
2. 4
3. $\sqrt{38}$.

$$\nabla f = (3x^2 - y, -x, 2z), \quad \nabla f(p) = (1, -1, 6), \quad |\nabla f(p)| = \sqrt{38}$$

Chain rule

$$g \circ f \colon \mathbb{R}^n \xrightarrow{f} \mathbb{R}^k \xrightarrow{g} \mathbb{R}^m$$

$$J(g \circ f)(p) = Jg(f(p))Jf(p)$$

$$D(g \circ f)_p \colon \mathbb{R}^n \xrightarrow{Df_p} \mathbb{R}^k \xrightarrow{Dg_{f(p)}} \mathbb{R}^m$$

Inverse function theorem

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

If $Jf(p)$ is invertible, then f is invertible in a neighborhood of p.

Thanks!