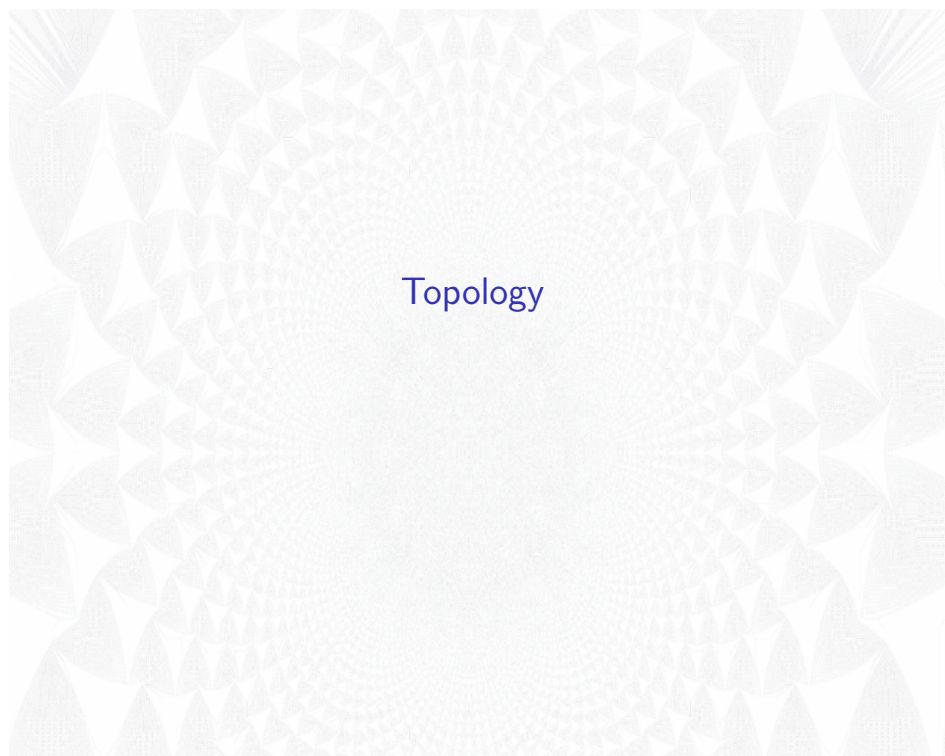




Topology

Definition

A *topology* on a set X is a collection of subsets τ of X such that

1. $\emptyset \in \tau$.
 2. $X \in \tau$.
 3. τ is closed under arbitrary unions.
 4. τ is closed under finite intersections.
- ▶ **topological space:** (X, τ) or just X .
 - ▶ **open sets** of the topology: τ .
 - ▶ **neighborhood of $x \in X$:** any set containing an open set containing x .
 - ▶ **$A \subseteq X$ closed:** the complement, A^c , is open.
 - ▶ **closure of $A \subseteq X$:** $\overline{A}$ = the intersection of all closed sets containing A .

Bases

A collection of subsets $\mathcal{B}$ of a set X is a **basis** for a topology on X if

1. $\mathcal{B}$ covers X .
2. if $x \in B' \cap B''$ for some $B', B'' \in \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$.

The topology **generated** by a basis $\mathcal{B}$ consists of all unions of sets in $\mathcal{B}$.

Open balls $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$ form a basis for the **standard topology** on $\mathbb{R}^n$.

second countable: $\exists$ countable basis

Continuous functions

$f: X \rightarrow Y$ is **continuous** if $f^{-1}(U) \subseteq X$ is open for all open $U \subseteq Y$.

It suffices to check that $f^{-1}(U)$ is open for all U in a basis or that $f^{-1}(C)$ is closed for all closed $C \subseteq Y$.

$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if and only if it is ε - δ continuous.

Homeomorphisms

$f: X \rightarrow Y$ is a **homeomorphism** if f is a continuous bijection and f^{-1} is continuous.

New spaces from old

X, Y be topological spaces

subspace topology on $A \subseteq X$: $\{U \cap A : U \text{ open in } X\}$

product topology on $X \times Y$: basis
 $\{U \times V : U, V \text{ open in } X, Y, \text{ respectively}\}$

quotient topology: given surjection $f: X \rightarrow Z$, let $U \subseteq Z$ be open iff $f^{-1}(U)$ is open in X

Topological property

A property that is preserved under continuous mappings is called a **topological property**.

Examples:

- ▶ Hausdorffness
- ▶ connectedness
- ▶ but *not* openness!

Hausdorff property

X is **Hausdorff** if for all $x, y \in X$ with $x \neq y$, there exists disjoint open neighborhoods U, V of x, y , respectively.

$\mathbb{R}^n$ is Hausdorff.

The **Zariski topology** in algebraic geometry is not Hausdorff.

Connectedness

X is **connected** if its only subsets that are both open and closed are $\emptyset$ and X .

X is **path connected** if for all $x, y \in X$, there exists a continuous path $f: [0, 1] \rightarrow X$ with $f(0) = x$ and $f(1) = y$.

Path connected $\Rightarrow$ connected.

If X is locally path connected, then the converse holds (e.g., manifolds).

Quiz (1/3)

Let $X = \{a, b, c, d\}$ with topology

$$\tau = \{\emptyset, X, \{a, b\}, \{a, b, d\}, \{b\}, \{b, d\}, \{b, c, d\}\}$$

Question 1. Is $\{b\}$ closed?

1. True
2. False

Solution: False since $\{b\}^c = \{a, c, d\} \notin \tau$.

Quiz (2/3)

Let $X = \{a, b, c, d\}$ with topology

$$\tau = \{\emptyset, X, \{a, b\}, \{a, b, d\}, \{b\}, \{b, d\}, \{b, c, d\}\}$$

Question 2. Is X Hausdorff?

1. True
2. False

Solution: False since, for example, there are no disjoint open sets separating a and b .

Quiz (3/3)

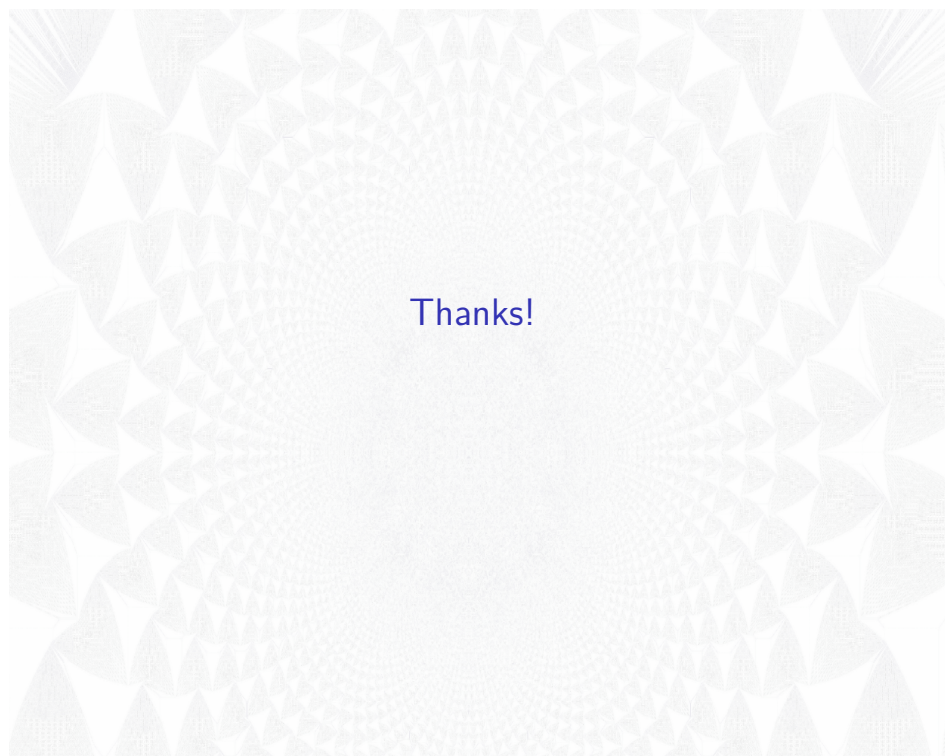
Let $X = \{a, b, c, d\}$ with topology

$$\tau = \{\emptyset, X, \{a, b\}, \{a, b, d\}, \{b\}, \{b, d\}, \{b, c, d\}\}$$

Question 3. Is X connected?

1. True
2. False

Solution: True: besides $\emptyset$ and X , there is no open set whose complement is also open. An easy way to see this is to note that every nonempty open set contains b . The complement of any nonempty open set will not contain b .



Thanks!