Topology

Definition

A *topology* on a set X is a collection of subsets τ of X such that

1. $\emptyset \in \tau$.
2. $X \in \tau$.
3. τ is closed under arbitrary unions.
4. τ is closed under finite intersections.

- topological space: (X, τ) or just X.
- open sets of the topology: τ.
- neighborhood of $x \in X$: any set containing an open set containing x.
- $A \subseteq X$ closed: the complement, A^c, is open.
- closure of $A \subseteq X$: $\overline{A} =$ the intersection of all closed sets containing A.

Bases

A collection of subsets $\mathcal{B}$ of a set X is a basis for a topology on X if

1. $\mathcal{B}$ covers X.
2. if $x \in B' \cap B''$ for some $B', B'' \in \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$.

The topology generated by a basis $\mathcal{B}$ consists of all unions of sets in $\mathcal{B}$.

Open balls $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$ form a basis for the standard topology on $\mathbb{R}^n$.

second countable: $\exists$ countable basis

Continuous functions

$f\colon X \to Y$ is continuous if $f^{-1}(U) \subseteq X$ is open for all open $U \subseteq Y$.

It suffices to check that $f^{-1}(U)$ is open for all U in a basis or that $f^{-1}(C)$ is closed for all closed $C \subseteq Y$.

$f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only it is ε-δ continuous.

Homeomorphisms

$f : X \to Y$ is a homeomorphism if f is a continuous bijection and f^{-1} is continuous.

New spaces from old

X, Y be topological spaces

subspace topology on $A \subseteq X$: $\{U \cap A : U \text{ open in } X\}$

product topology on $X \times Y$: basis
$\{U \times V : U, V \text{ open in } X, Y, \text{ respectively}\}$

quotient topology: given surjection $f : X \to Z$, let $U \subseteq Z$ be open iff $f^{-1}(U)$ is open in X

Topological property

A property that is preserved under continuous mappings is called a topological property.

Examples:

- Hausdorffness
- connectedness
- but *not* openness!

Hausdorff property

X is Hausdorff if for all $x, y \in X$ with $x \neq y$, there exists disjoint open neighborhoods U, V of x, y, respectively.

$\mathbb{R}^n$ is Hausdorff.

The Zarisksi topology in algebraic geometry is not Hausdorff.

Connectedness

X is connected if its only subsets that are both open and closed are $\emptyset$ and X.

X is path connected if for all $x, y \in X$, there exists a continuous path $f\colon [0,1] \to X$ with $f(0) = x$ and $f(1) = y$.

Path connected $\Rightarrow$ connected.

If X is locally path connected, then the converse holds (e.g., manifolds).

Quiz (1/3)

Let $X = \{a, b, c, d\}$ with topology
$\tau = \{\emptyset, X, \{a, b\}, \{a, b, d\}, \{b\}, \{b, d\}, \{b, c, d\}\}$

Question 1. Is $\{b\}$ closed?

1. True
2. False

Solution: False since $\{b\}^c = \{a, c, d\} \notin \tau$.

Quiz (2/3)

Let $X = \{a, b, c, d\}$ with topology
$\tau = \{\emptyset, X, \{a, b\}, \{a, b, d\}, \{b\}, \{b, d\}, \{b, c, d\}\}$

Question 2. Is X Hausdorff?

1. True
2. False

Solution: False since, for example, there are no disjoint open sets separating a and b.

Quiz (3/3)

Let $X = \{a, b, c, d\}$ with topology
$\tau = \{\emptyset, X, \{a, b\}, \{a, b, d\}, \{b\}, \{b, d\}, \{b, c, d\}\}$

Question 3. Is X connected?

1. True
2. False

Solution: True: besides $\emptyset$ and X, there is no open set whose complement is also open. An easy way to see this is is to note that every nonempty open set contains b. The complement of any nonempty open set will not contain b.

Thanks!