

NOTES ON MANIFOLDS

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CONTENTS

1. Introduction and Overview	3
2. Definition of a Manifold	6
2.1. Projective space	7
3. Differentiable Maps	10
4. Tangent Spaces	12
4.1. Three definitions of the tangent space	13
4.2. The three definitions are equivalent	18
4.3. Standard bases	22
4.4. The differential of a mapping of manifolds	22
5. Linear Algebra	25
5.1. Products and coproducts	25
5.2. Tensor Products	26
5.3. Symmetric and exterior products	29
5.4. Dual spaces	32
6. Vector Bundles	35
7. Tangent Bundle	38
8. The Algebra of Differential Forms	41
8.1. The pullback of a differential form by a smooth map	42
8.2. The exterior derivative	44
9. Oriented Manifolds	47
10. Integration of Forms	50
10.1. Definition and properties of the integral	50
10.2. Manifolds with boundary	52
10.3. Stokes' theorem on manifolds	56
11. de Rham Cohomology	58
11.1. Definition and first properties	58
11.2. Homotopy invariance of de Rham cohomology	60
11.3. The Mayer-Vietoris sequence	67
12. Differential Forms on Riemannian Manifolds	71
12.1. Scalar products	71
12.2. The star operator	73
12.3. Poincaré duality	75
13. Toric Varieties	79

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13.1.	Toric varieties from fans.....	79
13.2.	Toric varieties from polytopes.....	85
13.3.	Cohomology of toric varieties.....	85
13.4.	Homogeneous coordinates.....	87
13.5.	Mapping toric varieties into projective spaces.....	91
14.	Grassmannians.....	95
14.1.	Manifold structure.....	95
14.2.	The Plücker embedding.....	98
14.3.	Schubert varieties.....	100
Appendix A.	Vector Calculus in Euclidean Spaces.....	108
A.1.	Derivatives.....	108
A.2.	Classical integral vector calculus.....	109
Appendix B.	Topology.....	118
B.1.	Topological spaces.....	118
B.2.	Continuous functions and homeomorphisms.....	119
B.3.	Connectedness.....	120
B.4.	Compactness.....	120
B.5.	Partition of unity.....	121
Appendix C.	Measure Theory.....	121
Appendix D.	Simplicial homology.....	123
D.1.	Exact sequences.....	123
D.2.	Simplicial complexes.....	125
References		136

1. INTRODUCTION AND OVERVIEW

Differentiable manifolds are objects upon which one may do calculus without coordinates. They abstract “differential structure” just as vector spaces abstract linear structure. In both of these settings, the fundamental example is ordinary Euclidean space. Manifolds are much more complicated, though, as one might expect. The *only* difference between $\mathbb{R}^n$ and an n -dimensional vector space (over $\mathbb{R}$) in terms of linear structure is a choice of basis. Therefore, the natural numbers effectively classify vector spaces. In contrast, the classification of manifolds is a rich subject with many open problems.

Whereas an n -dimensional vector space is isomorphic to $\mathbb{R}^n$, a manifold is by its very definition *locally* isomorphic to an open subset of $\mathbb{R}^n$. That property is in accordance with what you know about differentiation in $\mathbb{R}^n$. It is a local process: its resulting value at any point only depends on the behavior of the function in question in any small neighborhood of the point. The structure of a manifold includes instructions for gluing together this local information.

There is no way to make measurements of distances and angles in a vector space until we add an inner product. An n -dimensional manifold M comes with a *tangent bundle* TM . It attaches a copy of $\mathbb{R}^n$ to each point (the *tangent space* at that point), and has the structure of a manifold, itself. In order to make measurements on a manifold, we need the additional structure of an inner product on the tangent space at each point, which varies smoothly with the point, resulting in a *metric*. Thus, our notion of distance will vary from point-to-point on a manifold. To see the utility of this notion, recall that mass distorts distances in our universe. A manifold with a metric is called a *Riemannian manifold*.

First example. We now describe the manifold structure of a two-dimensional sphere. While reading the following, please refer to Figure 1. Imagine the sphere as the surface of the earth. To find your way around on the earth, it suffices to have a world atlas. Each page h of the atlas is a chart representing some portion of the earth. The page is essentially a mapping $h: U \rightarrow U'$ from some region U of the earth to a rectangle U' in $\mathbb{R}^2$. Using the features of the earth, one can infer the nature of h by just looking at its image, i.e., the actual page in the atlas. From now on, we will identify these two things. Let $k: V \rightarrow V'$ be another chart/page. Suppose the regions U and V on the earth meet with overlap $W := U \cap V$. That means images of W will appear on the two pages we are considering. Since drawing the earth on flat paper requires stretching, these two images will be distorted copies of each other. However, assuming your atlas has sufficient details, we can tell which points on the two images represent the same point on the earth.

Imagine now that the pages of your atlas are made of malleable putty, and your job is to reconstruct the earth. You cut out all the pages and are left with stretching and gluing together the overlaps. The likely result, given the nature of putty, will be a lop-sided lumpy version of a sphere.

In extracting the manifold structure from this example, it is important to remember that a manifold does not come with a metric. So we should try to forget

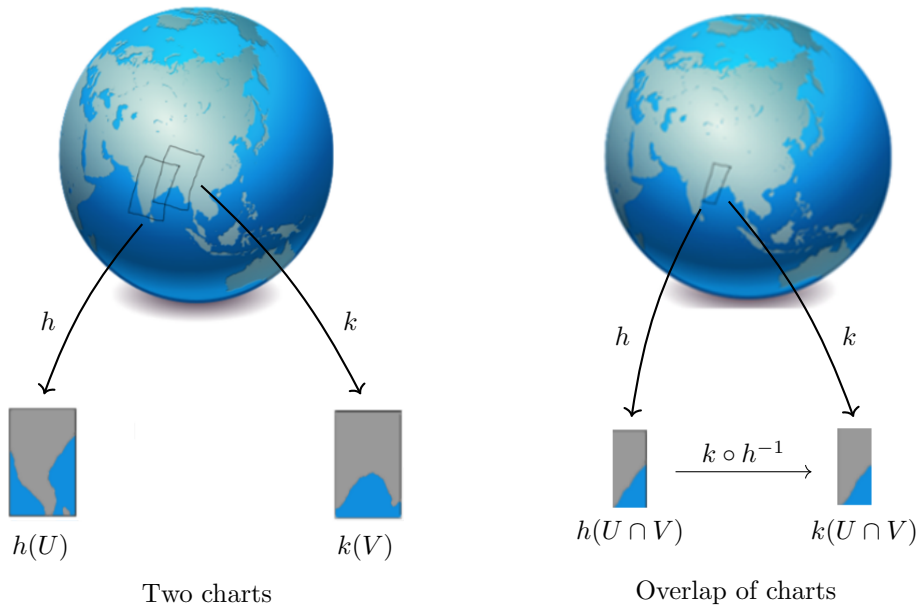


FIGURE 1. Charts (U, h) and (V, k) on the sphere and their corresponding transition mapping on the overlap.

that aspect of an ordinary atlas of the earth. Instead, as above, think of the underlying substrate of a manifold as putty. In mathematical terms, the pages of the atlas are open subsets of a topological space. We will assume the reader can quickly “review” the features of topology summarized in Appendix B.

One last observation: unlike our example, a manifold does not come with an embedding into Euclidean space—the embedding is separate information. Whitney’s embedding theorem says that, in general, an n -dimensional manifold can be embedded in $\mathbb{R}^{2n}$. So surfaces (two-dimensional manifolds) can be embedded in $\mathbb{R}^4$, but something special needs to occur to get an embedding in $\mathbb{R}^3$, as in the case of a sphere. The Klein bottle, which cannot be embedded in $\mathbb{R}^3$, is more typical.

Integration. While differentiation is a local process, integration is a global process. Thinking of our manifold as being created by gluing together stretchable pieces of putty, it is clear that integration will require adding some structure to the manifold. Having chosen coordinates near a point, we can integrate functions as usual, but what if we change coordinates? Of course, there is the usual change of coordinates formula from several-variable calculus, but how would we choose the initial set of coordinates (from which we could change)? And even more perplexing: what if we want to integrate over a portion of the manifold that is not contained in a single chart? These questions suggest that in order to perform integration on a manifold, we will need a structure that oversees these changes of coordinates and provides a standard of some sort, against which we can measure. That structure is called a non-vanishing n -form and will take us a while to define. Once we have it, though,

one of our goals will be to prove the ultimate version of Stokes' theorem:

$$\int_{\partial M} \omega = \int_M d\omega.$$

Here, ω is an n -form and ∂M is the “boundary” of M (so we will need to consider manifolds with boundaries). The operator d is called the *exterior derivative*. It will have the property that applying d twice gives $d^2 = 0$, which may remind you of some results from ordinary vector calculus in $\mathbb{R}^n$. Exploring this property leads to remarkable topological invariants in the form of cohomology groups. If time permits, we will consider these groups in the case of two classes of manifolds: toric varieties and Grassmannians. The cohomology of Grassmann manifolds has a ring structure (i.e., well-behaved addition and multiplication) known as the *Schubert calculus*.

2. DEFINITION OF A MANIFOLD

Definition 2.1 (Charts). Let X be a topological space. An n -dimensional chart on X is a homeomorphism $h: U \xrightarrow{\cong} U'$ from an open subset $U \subseteq X$, the *chart domain*, onto an open subset $U' \subseteq \mathbb{R}^n$. We denote this chart by (U, h) (see Figure 2). We say that X is *locally Euclidean* if every point in X belongs to some chart domain of X . If X is locally Euclidean, choosing a chart containing some point $p \in X$ is called *taking local coordinates at p* .

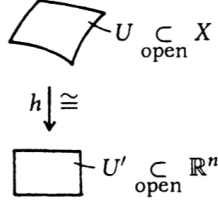


FIGURE 2. A chart.

Definition 2.2. If (U, h) and (V, k) are two n -dimensional charts on X such that $U \cap V \neq \emptyset$, then the homeomorphism $(k \circ h^{-1})|_{h(U \cap V)}$ from $h(U \cap V)$ to $k(U \cap V)$ is called the *change-of-charts map*, *change of coordinates*, or *transition map*, from h to k (see Figure 3). If the transition map is furthermore a *diffeomorphism* (note that

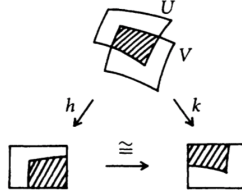


FIGURE 3. Transition map.

these maps have domains and codomains in $\mathbb{R}^n$), then we say that the two charts are *differentially related*. Recall that a function between subsets of Euclidean spaces is a *diffeomorphism* if it is bijective and both it and its inverse are differentiable. Throughout this text, we will take *differentiable* to mean *smooth*, i.e., having partial derivatives of all orders.

Definition 2.3. A set of n -dimensional charts on X whose chart domains cover all of X is an n -dimensional *atlas* on X . The atlas is *differentiable* if all its charts are differentially related, and two differentiable atlases $\mathfrak{A}$ and $\mathfrak{B}$ are *equivalent* if $\mathfrak{A} \cup \mathfrak{B}$ is also differentiable.

Definition 2.4. An n -dimensional *differentiable structure* on a topological space X is a maximal n -dimensional differentiable atlas with respect to inclusion.

Definition 2.5 (Differentiable manifolds). An n -dimensional differentiable manifold¹ is a pair $(M, \mathcal{D})$ consisting of a second countable Hausdorff topological space M with an n -dimensional differentiable structure $\mathcal{D}$.

Example 2.6. As a first (trivial) example of an n -manifold, take any open subset $U \subseteq \mathbb{R}^n$ with the atlas $\{(U, \text{id}_U)\}$ containing a single chart.

Example 2.7 (The n -sphere). The n -sphere is the set

$$S^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

Thus, for example, the one-sphere is the unit circle in the plane, and the two-sphere is the usual sphere in three-space. If $p = (p_1, \dots, p_{n+1}) \in S^n$, then there exists some i such that $p_i \neq 0$. Let U be any open neighborhood of p consisting solely of points on the sphere whose i -coordinates are nonzero. Then the mapping at $h: U \rightarrow h(U) \subset \mathbb{R}^n$ defined by dropping the i -th coordinate of each point in U serves as a chart containing p .

For another atlas, compatible with the one just given, one can use *stereographic projection*. Here the atlas has two open sets:

$$U^+ := S^n \setminus \{(0, 0, \dots, 0, 1)\} \quad \text{and} \quad U^- := S^n \setminus \{(0, 0, \dots, 0, -1)\}$$

each with chart defined by

$$(x_1, \dots, x_{n+1}) \mapsto \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right).$$

2.1. Projective space. *Projective n -space*, denoted $\mathbb{P}^n$, is an n -manifold whose points are the lines in $\mathbb{R}^{n+1}$ passing through the origin, i.e., the collection of all one-dimensional linear subspaces of $\mathbb{R}^{n+1}$. To represent a line ℓ through the origin in $\mathbb{R}^{n+1}$, choose any nonzero point $p \in \ell$. (The point p is a basis for ℓ as a one-dimensional linear subspace.) For points $p, q \in \mathbb{R}^{n+1}$ write $p \sim q$ if $p = \lambda q$ for some nonzero constant λ . Then p and q represent the same line if and only if $p \sim q$. So we take our formal definition of projective n -space to be

$$\mathbb{P}^n := \mathbb{R}^{n+1} \setminus \{\vec{0}\} / \sim$$

with the quotient topology. (Thus, a set of points in $\mathbb{P}^n$ is open if and only if the union of the set of lines they represent is an open subset of $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$.)

If $\ell \in \mathbb{P}^n$ is represented by (the equivalence class of) $p = (a_1, \dots, a_{n+1}) \in \ell$, then $(a_1, \dots, a_{n+1})$ are called the *homogeneous coordinates* for ℓ , realizing that these “coordinates” are only defined up to scaling by a nonzero constant. One sometimes sees the notation $(a_1 : a_2 : \dots : a_{n+1})$ to represent the point $\ell \in \mathbb{P}^n$, but we will stick with $(a_1, \dots, a_{n+1})$.

We would now like to impose a manifold structure on $\mathbb{P}^n$. As a warm-up, we first treat the case $n = 2$. Define the set

$$\begin{aligned} U_x &:= \{(x, y, z) \in \mathbb{P}^2 : x \neq 0\} \\ U_y &:= \{(x, y, z) \in \mathbb{P}^2 : y \neq 0\} \end{aligned}$$

¹We will typically refer to these simply as n -manifolds.

$$U_z := \{(x, y, z) \in \mathbb{P}^2 : z \neq 0\}.$$

Note the following:

- (1) These sets are well-defined. Even though homogeneous coordinates are only defined up to a nonzero scalar constant. Nevertheless, replacing (x, y, z) by $\lambda(x, y, z)$ with $\lambda \neq 0$, we have $x = 0$ if and only if $\lambda x = 0$, and similarly for y and z .
- (2) These are open sets. Since we are using the quotient topology, we need to consider $\pi^{-1}(U_x)$ which is the complement of the y, z plane in $\mathbb{R}^3 \setminus \{\vec{0}\}$, which is open in $\mathbb{R}^3 \setminus \{\vec{0}\}$. A similar argument holds for U_y and U_z .
- (3) These sets cover $\mathbb{P}^2$. If $(x, y, z) \in \mathbb{P}^2$, then at least one of x, y , or z must be nonzero.

Finally, we define the chart mappings:

$$\begin{array}{lll} \phi_x: U_x \rightarrow \mathbb{R}^2 & \phi_y: U_y \rightarrow \mathbb{R}^2 & \phi_z: U_z \rightarrow \mathbb{R}^2 \\ (x, y, z) \mapsto (y/x, z/x) & (x, y, z) \mapsto (x/y, z/y) & (x, y, z) \mapsto (x/z, y/z) \end{array}$$

It is easy to check that each of these is a homeomorphism. For instance, the inverse of ϕ_x is given by $(u, v) \mapsto (1, u, v)$. The underlying motivation for these charts is as follows: Take U_x , for instance. Fix the plane $x = 1$ in $\mathbb{R}^3$. Then a line through the origin in $\mathbb{R}^3$ meets this plane if and only if a representative nonzero point on the line has x -coordinate not equal to 0. So U_x consists of the lines meeting the plane $x = 1$. If $\ell \in U_x$ has homogeneous coordinates (x, y, z) , we can scale (x, y, z) by $\lambda = 1/x$ to get another representative $(1, y/x, z/x)$. This is exactly the point where ℓ meets the plane $x = 1$. In this way, each point in U_x has a set of homogeneous coordinates of the form $(1, u, v)$. Dropping the 1, which is superfluous information, gives us the mapping ϕ_x . The essential idea is that there is a one-to-one correspondence between points in the plane $x = 1$ and points in U_x , and the plane $x = 1$ is the same as $\mathbb{R}^2$ via $(1, u, v) \mapsto (u, v)$.

The collection

$$\{(U_x, \phi_x), (U_y, \phi_y), (U_z, \phi_z)\}$$

is the *standard atlas* for $\mathbb{P}^2$. What does a typical transition function look like? Consider the transition from U_x to U_y . The overlap is $U_x \cap U_y = \{(x, y, z) \in \mathbb{P}^2 : x \neq 0, y \neq 0\}$, and we have the commutative diagram:

$$\begin{array}{ccc} & (1, u, v) & \\ \phi_x^{-1} \swarrow & & \searrow \phi_y \\ (u, v) & \xrightarrow{\phi_y \circ \phi_x^{-1}} & (1/u, v/u). \end{array}$$

In general, the *standard charts* of $\mathbb{P}^n$ are given by (U_i, ϕ_i) for $i = 1, \dots, n+1$, where $U_i = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{P}^n \mid x_i \neq 0\}$, and

$$\begin{aligned} \phi_i: U_i &\rightarrow \mathbb{R}^n \\ x &\mapsto \left(\frac{x_1}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right). \end{aligned}$$

The hat over $\frac{x_i}{x_i}$ means that this component of the vector should be omitted (just like we did for $\mathbb{P}^2$).

3. DIFFERENTIABLE MAPS

Let M be a manifold and X some topological space. To study the behavior of a map $f: M \rightarrow X$ near some point $p \in M$, we choose a chart (U, h) at p and look at the “downstairs map” $f \circ h^{-1}: h(U) \rightarrow X$ instead (see Figure 4). If $f \circ h^{-1}$ has a

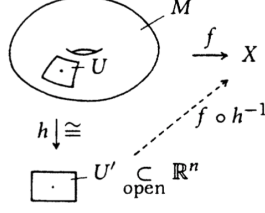


FIGURE 4. The downstairs map $f \circ h^{-1}$.

certain property locally at $h(p)$, then we say that f has the property at p *relative to the chart* (U, h) . If this property is independent of the choice of charts, then we just say that f has this property at p . Our first example of such a property is the *differentiability* of a function.

Definition 3.1 (Real-valued functions on M). A function $f: M \rightarrow \mathbb{R}$ is *differentiable at* $p \in M$ if $f \circ h^{-1}$ is differentiable for some chart (U, h) at p .

Exercise 3.2. Let $f: M \rightarrow \mathbb{R}$ be a function on a manifold M , and let (U, h) and (V, k) be two charts at $p \in M$. Show that if f is differentiable at p relative to (U, h) , then f is differentiable at p relative to (V, k) . (You can use the fact that a composition of differentiable functions on Euclidean space is differentiable.)

Definition 3.3 (Differentiable mappings of manifolds). A continuous map $f: M \rightarrow N$ between manifolds is *differentiable at* $p \in M$ if it is differentiable with respect to charts at $p \in M$ and at $f(p) \in N$. Namely, if (U, h) is a chart at p and (V, k) is a chart at $f(p)$ such that $f(U) \subseteq V$, we want the map $k \circ f \circ h^{-1}$ to be differentiable (recall that $h(U) \subseteq \mathbb{R}^m$ and $k(V) \subseteq \mathbb{R}^n$ for some m, n):

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ h \downarrow & & \downarrow k \\ h(U) & \xrightarrow[k \circ f \circ h^{-1}]{} & k(V) \end{array}$$

For a picture of this, see Figure 5. If f is bijective with a differentiable inverse, then f is called a *diffeomorphism*.

Remark 3.4. The reader should check that differentiability at $p \in M$ is independent of the choice of charts.

Example 3.5 (The Veronese embedding). Define $\nu_2: \mathbb{P}^2 \rightarrow \mathbb{P}^5$ by

$$\begin{aligned} \nu_2: \mathbb{P}^2 &\rightarrow \mathbb{P}^5 \\ (x, y, z) &\mapsto (x^2, xy, xz, y^2, yz, z^2). \end{aligned}$$

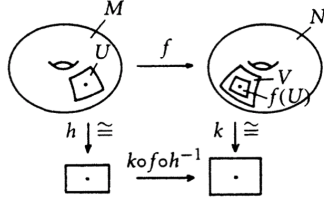


FIGURE 5. Using the charts to pull down a continuous map between manifolds.

This mapping is well-defined since

$$\nu_2(\lambda(x, y, z)) = \nu_2(\lambda x, \lambda y, \lambda z) = \lambda^2(x^2, xy, xz, y^2, yz, z^2) = \nu_2(x, y, z).$$

We would like to show that ν_2 is differentiable. Let $p = (a, b, c) \in \mathbb{P}^2$, and without loss of generality, suppose that $a \neq 0$. Let (U_x, ϕ_x) be the standard chart at p whose domain consists of points with nonzero first coordinates. Then $\nu_2(p) \in \nu_2(U_x) \subset U_1 \subset \mathbb{P}^5$, and so we take the standard chart (U_1, ϕ_1) at $\nu_2(p)$. Using these charts, we calculate

$$\phi_1 \circ \nu_2 \circ \phi_x^{-1}(u, v) = \phi_1(\nu_2(1, u, v)) = \phi_1(1, u, v, u^2, uv, v^2) = (u, v, u^2, uv, v^2),$$

which is differentiable.

Example 3.6. The existence of a diffeomorphism between two manifolds means that as far as manifold structures are concerned, there is no difference between the manifolds (except for, possibly, their names). Thinking in those terms, the following theorem may be surprising:

Theorem 3.7 (Milnor). *There exist differentiable structures $\mathcal{D}$ and $\mathcal{D}'$ on the seven-sphere S^7 with no diffeomorphism $(S^7, \mathcal{D}) \rightarrow (S^7, \mathcal{D}')$.*

4. TANGENT SPACES

If we try to imagine a typical tangent space, we might think of a surface S sitting inside $\mathbb{R}^3$ with a plane “kissing” the surface at some point. To compute the tangent plane, we could create a parametrization of the surface near that point. This would amount to finding an open set $U \subseteq \mathbb{R}^2$ and a “nice”² differentiable mapping $f: U \rightarrow \mathbb{R}^3$ with a point $p \in U$ mapping to the point in question, $f(p)$. Recall that by definition,

$$\lim_{|h| \rightarrow 0} \frac{|f(p+h) - f(p) - Df_p(h)|}{|h|} = 0$$

where $Df_p: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the derivative of f at p . Hence, *up to first order*, we have $f(p+h) = f(p) + Df_p(h)$ where Df_p is the derivative of f at p . Letting $Af_p(h) := f(p+h)$, we get the *best affine approximation to f at p* :

$$\begin{aligned} Af_p: \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \\ h &\mapsto f(p) + Df_p(h). \end{aligned}$$

Its image will be the tangent plane at $f(p)$.

An underlying assumption we made above in thinking about the tangent space is that our surface and its tangent plane are embedded in a larger space, $\mathbb{R}^3$. Our goal in this section is to take on the intriguing task of creating an *intrinsic* definition of the tangent space of a manifold at a given point, i.e., one that does not depend on embedding the manifold into another space. We will give three constructions of the tangent space which we will call the *geometric*, the *algebraic*, and the *physical* definitions of the tangent space, and we will see that all three are equivalent.³

At a couple of places in the following discussion, we will need a technical lemma. We’ll get that out of the way now:

Lemma 4.1. *Let $f: W \rightarrow \mathbb{R}$ be a smooth function on some open subset W of $\mathbb{R}^n$, and let $w \in W$. Then there exist smooth functions $g_i: W \rightarrow \mathbb{R}$ for $i = 1, \dots, n$ satisfying:*

$$g_i(p) = \frac{\partial f}{\partial x_i}(p) \quad \text{and} \quad f(x) = f(p) + \sum_{i=1}^n g_i(x)(x_i - p_i).$$

Proof. For $x \in W$, apply the fundamental theorem of calculus and then the chain rule to get

$$\begin{aligned} f(x) - f(p) &= \int_0^1 \frac{d}{dt} f(tx + (1-t)p) dt \\ &= \int_0^1 \sum_{i=1}^n \left[\frac{\partial f}{\partial x_i}(tx + (1-t)p) \right] (x_i - p_i) dt \\ &= \sum_{i=1}^n \left[\int_0^1 \frac{\partial f}{\partial x_i}(tx + (1-t)p) dt \right] (x_i - p_i). \end{aligned}$$

²“Nice” would mean that the mapping and its derivative are injective on U .

³These names are ad hoc. There is no standard terminology.

Define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx + (1-t)p) dt.$$

□

For the remaining of this section, let M be a manifold and take $p \in M$.

4.1. Three definitions of the tangent space.

Definition 4.2 (Geometrically-defined tangent space). Let $\mathcal{K}_p(M)$ denote the set of differentiable curves in M that pass through p at 0. More precisely,

$$\mathcal{K}_p(M) = \{\alpha: (-\varepsilon, \varepsilon) \rightarrow M \mid \alpha \text{ is differentiable, } \varepsilon > 0, \text{ and } \alpha(0) = p\}.$$

Two such curves $\alpha, \beta \in \mathcal{K}_p(M)$ are called *tangentially equivalent*, denoted $\alpha \sim \beta$, if

$$(h \circ \alpha)'(0) = (h \circ \beta)'(0) \in \mathbb{R}^n$$

for some (hence any) chart (U, h) around p . We call the equivalence classes $\mathcal{K}_p(M)$ the (*geometrically-defined*) *tangent vectors* of M at p and call the quotient

$$T_p^{\text{geom}} M := \mathcal{K}_p(M) / \sim$$

the (*geometrically-defined*) *tangent space* to M at p .

Exercise 4.3. Check that $\sim$ is independent of the choice of chart (U, h) .

Now let us put a linear structure on $T_p^{\text{geom}} M$. Fix a chart (U, h) at p and define

$$\begin{aligned} \phi: T_p^{\text{geom}} M &\rightarrow \mathbb{R}^n \\ [\alpha] &\mapsto (h \circ \alpha)'(0). \end{aligned}$$

Then ϕ is a bijection (check!). So we can use it to induce a linear structure on $T_p^{\text{geom}} M$. For curves $\alpha, \beta \in \mathcal{K}_p(M)$ and any real number λ , define

$$\lambda[\alpha] + [\beta] := \phi^{-1}(\lambda\phi(\alpha) + \phi(\beta)).$$

The result is independent of choice of chart: Let (V, k) be another chart at p and similarly define $\psi: T_p^{\text{geom}} M \rightarrow \mathbb{R}^n$ that sends $[\alpha]$ to $(k \circ \alpha)'(0)$. We need to show that for $\alpha, \beta \in \mathcal{K}_p(M)$, we have

$$\phi^{-1}(\lambda\phi(\alpha) + \phi(\beta)) = \psi^{-1}(\lambda\psi(\alpha) + \psi(\beta)).$$

Pick $\gamma \in \mathcal{K}_p(M)$ such that $[\gamma] = \phi^{-1}(\lambda\phi(\alpha) + \phi(\beta))$. Equivalently, applying the bijection ϕ to both sides of this equation,

$$(h \circ \gamma)'(0) = \lambda\phi(\alpha) + \phi(\beta).$$

Our goal is to show that $[\gamma] = \psi^{-1}(\lambda\psi(\alpha) + \psi(\beta))$, i.e., that

$$(k \circ \gamma)'(0) = \lambda\psi(\alpha) + \psi(\beta).$$

This follows from the chain rule⁴:

$$(k \circ \gamma)'(0) = J(k \circ \gamma)(0)$$

⁴In the following, as always, we identify vectors in $\mathbb{R}^n$ with column matrices.

$$\begin{aligned}
&= J(k \circ h^{-1} \circ h \circ \gamma)(0) \\
&= J(k \circ h^{-1})(h(p))J(h \circ \gamma)(0) \\
&= J(k \circ h^{-1})(h(p))(h \circ \gamma)'(0) \\
&= J(k \circ h^{-1})(h(p))(\lambda\phi(\alpha) + \phi(\beta)) \\
&= \lambda J(k \circ h^{-1})(h(p))J(h \circ \alpha)(0) + J(k \circ h^{-1})(h(p))J(h \circ \beta)(0) \\
&= \lambda J(k \circ h^{-1} \circ h \circ \alpha)(0) + J(k \circ h^{-1} \circ h \circ \beta)(0) \\
&= \lambda(k \circ \alpha)'(0) + (k \circ \beta)'(0) \\
&= \lambda\psi(\alpha) + \psi(\beta).
\end{aligned}$$

By definition of the linear structure on $T_p^{\text{geom}}(M)$, the mapping ϕ is an isomorphism of vector spaces. Thus, $\dim T_p^{\text{geom}}(M) = n$. The zero vector for $T_p^{\text{geom}}(M)$ is the class of the constant curve at $t \mapsto p$ for all $t \in (-1, 1)$.

We now move on to the *algebraic* version of the tangent space. For a moment, let's think about ordinary vector calculus in $\mathbb{R}^n$, i.e., the case $M = \mathbb{R}^n$. Let $p \in \mathbb{R}^n$, and take a vector $v \in \mathbb{R}^n$. Think of v as a tangent vector at p . Then given a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, we could ask how fast f is changing along v . In other words we could compute

$$f_v(p) := \lim_{t \rightarrow 0} \frac{f(p + tv) - f(p)}{t}.$$

[Check that $f_v(p) = \nabla f(p) \cdot v$.] If v is a unit vector, this would be the directional derivative of f in the direction of v . What are the main properties of the derivative? For one, the derivative should be a linear function. In addition, its main property is the product rule (also known as the Leibniz rule).

We want to use this idea of a directional derivative to define tangent vectors on an arbitrary manifold. (It does not carry over verbatim since, for instance, $p + tv$ would not make sense when p is a point in a general manifold.) The idea is that a tangent vector at a point p should give us a kind of derivative of real-valued functions defined near p . (The function does not need to be defined on the whole manifold because, after all, the derivative at a point is a local property.) Thus, we are led to the idea that a tangent vector at p should be a linear function on the space of functions defined near p , and that linear function should satisfy the product rule. We now make these notions precise.

Definition 4.4. Let two real-valued functions, each defined and differentiable in some neighborhood of a point p of M , be called *equivalent* if they agree in a neighborhood of p . The equivalence classes are called the *germs of differentiable functions on M at p* , and the set of these germs is denoted by $\mathcal{E}_p(M)$.

Remark 4.5. Note that equivalent functions need not be the same on the entire intersection of their domains, as illustrated in Figure 6.

Remark 4.6. The collection of germs $\mathcal{E}_p(M)$ is more than just a vector space—it is an $\mathbb{R}$ -algebra! Namely, for $[f], [g] \in \mathcal{E}_p(M)$, apart from the usual operations of addition and scaling by $\mathbb{R}$, we can also multiply: $[f] \cdot [g] := [fg]$. (Check that this multiplication is well-defined.)

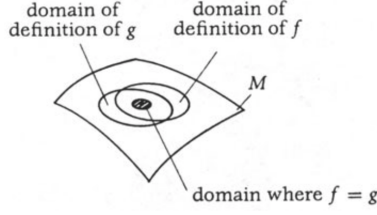


FIGURE 6. Equivalent functions do not need to agree on the entirety of their common domains.

Definition 4.7 (Algebraically-defined tangent space). By an (*algebraically-defined*) *tangent vector* to M at p , we mean a *derivation* of the ring $\mathcal{E}_p(M)$ of germs, that is, a linear map on the germs

$$v: \mathcal{E}_p(M) \rightarrow \mathbb{R}$$

that satisfies the product rule

$$v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g)$$

for all $f, g \in \mathcal{E}_p(M)$. We call the vector space of these derivations the (*algebraically-defined*) *tangent space* to M at p and denote it by $T_p^{\text{alg}}(M)$.

Exercise 4.8. Let $f: M \rightarrow \mathbb{R}$ be a constant function. Let v be a derivation of $\mathcal{E}_p(M)$. What is $v(f)$?

We would now like to describe a basis for $T_p^{\text{alg}}(M)$. Fix a chart (U, h) at p and define the derivations ∂_i for $i = 1, \dots, n$ by

$$\partial_i(f) := \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p))$$

for each $f \in \mathcal{E}_p(M)$. In other words, we use the chart (U, h) to identify f with an ordinary multivariable function, and then take its i -th partial derivative. We leave the straightforward check that each ∂_i is a derivation to the reader. The reader may also check that

$$\partial_i(h_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We claim that these ∂_i form a basis for T_p^{alg} . By the above displayed equation, they are linearly independent. To see they span is not so easy. So the reader may want to put off the following argument until well-rested! Take $v \in T_p^{\text{alg}}$, and let f be a smooth real-valued function defined near p . Define $\ell: h(U) \rightarrow \mathbb{R}$ by $\ell := f \circ h^{-1}$. By Lemma 4.1,

$$\begin{aligned} \ell(x) &= \ell(h(p)) + \sum_{i=1}^n g_i(x)(x_i - h(p)_i) \\ &= f(p) + \sum_{i=1}^n g_i(x)(x_i - h(p)_i) \end{aligned}$$

for some smooth functions g_i such that

$$g_i(h(p)) = \frac{\partial \ell}{\partial x_i}(h(p)) = \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p)).$$

Therefore,

$$f(x) = \ell(h(x)) = f(p) + \sum_{i=1}^n g_i(h(x))(h(x)_i - h(p)_i)$$

Using the fact that v is linear and satisfies the product rule, we get

$$\begin{aligned} v(f) &= v(f(p) + \sum_{i=1}^n g_i(h(x))(h(x)_i - h(p)_i)) \\ &= v(f(p)) + \sum_{i=1}^n v(g_i(h(x))(h(x)_i - h(p)_i)) \\ &= \sum_{i=1}^n v(g_i(h(x))(h(x)_i - h(p)_i)) \\ &= \sum_{i=1}^n g_i(h(p))v(h(x)_i - h(p)_i) + v(g_i(h(x))) \cdot 0 \\ &= \sum_{i=1}^n g_i(h(p))v(h(x)_i) \end{aligned}$$

Letting $\alpha_i := v(h(x)_i) \in \mathbb{R}$, we have

$$v(f) = \sum_{i=1}^n \alpha_i g_i(h(p)) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p)) = \sum_{i=1}^n \alpha_i \partial_i(f).$$

Thus,

$$v = \sum_{i=1}^n \alpha_i \partial_i,$$

as required.

For our last formulation of tangent space, we take perhaps the most straightforward approach. We would like to define the tangent space at $p \in M$ by choosing a chart, thus identifying M with $\mathbb{R}^n$ near p . We then take any vector $v \in \mathbb{R}^n$, and think of it as a tangent vector at p . The problem with this approach is that it would depend on a choice of charts, and the whole point of manifolds is to formulate calculus without coordinates. As a minimal fix then, let's repeat this process for every possible chart at p . Thus, we think of a tangent vector as being a collection of vectors in $\mathbb{R}^n$, one for each chart at p . However, these vectors should somehow reflect the way we glue charts together to construct the manifold, i.e., these vectors should satisfy some kind of compatibility condition as we change coordinates. From that point of view, it is perhaps natural to require the choice of vectors for each pair of charts to be related via the derivative of the transition function between the charts.

Definition 4.9 (Physically-defined tangent space). Let M be an n -dimensional manifold, $p \in M$. Let $\mathcal{D}_p(M) := \{(U, h) \in \mathcal{D} \mid p \in U\}$ denote the set of charts around p . By a (*physically-defined*) *tangent vector* v to M at p , we mean a mapping

$$v: \mathcal{D}_p(M) \rightarrow \mathbb{R}^n$$

with the property that for any two charts at p , the associated vectors in $\mathbb{R}^n$ are mapped to each other by the derivative of the transition map; that is,

$$v(V, k) = D_{h(p)}(k \circ h^{-1})(v(U, h))$$

for all $(U, h), (V, k) \in \mathcal{D}_p(M)$. We call the vector space of these maps v the (*physically-defined*) *tangent space* to M at p and denote it by $T_p^{\text{phy}} M$.

Remark 4.10. There is an explicit way to describe the property defining a physical tangent vector. Define $x^i = h_i$, the i -th component of h . Similarly, define $\tilde{x}^i = k_i$, $v^i = v(U, h)_i$, and $\tilde{v}^i(V, k)_i$. Referring to Figure 7, if we assume for convenience

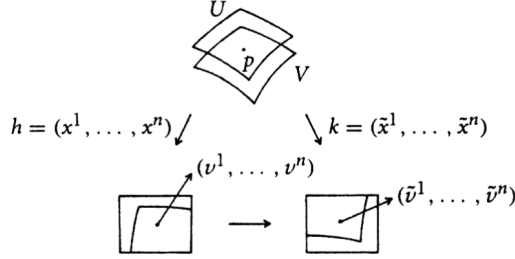


FIGURE 7. Physically-defined tangent vector in coordinates.

that $h(p) = 0$, in terms of the Jacobian matrix of $k \circ h^{-1}$, the equation $v(V, k) = D_{h(p)}(k \circ h^{-1})(v(U, h))$ becomes

$$[J(k \circ h^{-1})_0]v(U, h) = \left[\frac{\partial \tilde{x}^i}{\partial x^j} \right] \Big|_{x=0} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{bmatrix} = v(V, k),$$

where $\left[\frac{\partial \tilde{x}^i}{\partial x^j} \right]$ of partials of the components of $k \circ h^{-1}$.

To be explicit about the linear structure on $T_p^{\text{phy}}(M)$, if $v, w \in T_p^{\text{phy}}(M)$ and $\lambda \in \mathbb{R}$, then, by definition,

$$(\lambda v + w)(U, h) := \lambda v(U, h) + w(U, h)$$

for each chart (U, h) at p . Note that a physically-defined tangent vector $v: \mathcal{D}_p(M) \rightarrow \mathbb{R}^n$ is determined by its value on any particular chart (U, h) . Its value on any other chart (V, k) is then given by applying the derivative of the transition, as stated in Definition 4.9. In particular, fixing a chart (U, h) determines an isomorphism of vector spaces.

$$T_p^{\text{phy}}(M) \rightarrow \mathbb{R}^n$$

$$v \mapsto v(U, h).$$

4.2. The three definitions are equivalent. The following proposition shows that our three definitions of tangent space are just three perspectives on the same thing.

Proposition 4.11. *Let M be an n -dimensional manifold and let $p \in M$. There are canonical (i.e., do not involve choosing bases) linear isomorphisms Φ_1, Φ_2, Φ_3 that make the following diagram commute.*

$$\begin{array}{ccc} & T_p^{\text{geom}}(M) & \\ \Phi_3 \nearrow & & \searrow \Phi_1 \\ T_p^{\text{phy}}(M) & \xleftarrow{\Phi_2} & T_p^{\text{alg}}(M) \end{array}$$

Proof. We start by describing the maps Φ_1, Φ_2, Φ_3 .

1. $T_p^{\text{geom}}(M) \longrightarrow T_p^{\text{alg}}(M)$.

Define

$$\begin{aligned} \Phi_1: T_p^{\text{geom}}(M) &\longrightarrow T_p^{\text{alg}}(M) \\ [\alpha] &\longmapsto v_{[\alpha]}: [f] \mapsto (f \circ \alpha)'(0) \end{aligned}$$

where $\alpha: (-\varepsilon, \varepsilon) \rightarrow M$ is a differentiable curve on M with $\alpha(0) = p$ and $f \in \mathcal{E}_p(M)$ is a representative of a germ on M at p .

First, we check that this map is well-defined. If $[f] = [g] \in \mathcal{E}_p(M)$ for some g , then since f and g agree on some smaller neighborhood W of p , we have $f \circ \alpha = g \circ \alpha$ once we restrict to the appropriate domain. At the same time, if $\beta: (-\varepsilon, \varepsilon) \rightarrow M$ is a differentiable curve on M such that $\alpha \sim \beta$ (we can always restrict the domains so that they are the same for both α and β), then the chain rule tells us that $v_{[\alpha]} = v_{[\beta]}$. In detail, suppose that the images of α and of β are contained in some chart (U, h) at p (see Figure 8).

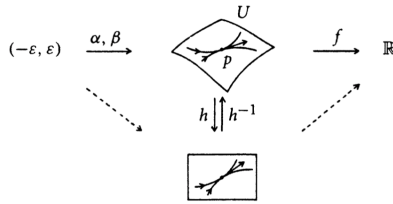


FIGURE 8. From $T_p^{\text{geom}}(M)$ to $T_p^{\text{alg}}(M)$.

Since $\alpha \sim \beta$, it follows that $(h \circ \alpha)'(0) = (h \circ \beta)'(0)$. So we have

$$\begin{aligned} (f \circ \alpha)'(0) &= J(f \circ h^{-1} \circ h \circ \alpha)(0) \\ &= J(f \circ h^{-1})(h(p))J(h \circ \alpha)(0) \\ &= J(f \circ h^{-1})'(h(p))J(h \circ \beta)(0) \end{aligned}$$

$$\begin{aligned}
&= J(f \circ h^{-1} \circ h \circ \beta)(0) \\
&= (f \circ \beta)'(0).
\end{aligned}$$

Next, we show that the image of Φ_1 is in $T_p^{\text{alg}}(M)$. Namely, $v_{[\alpha]}$ is a derivation. Let $g \in \mathcal{E}_p(M)$. Using the product rule, we have

$$\begin{aligned}
v_{[\alpha]}(f \cdot g) &= ((f \cdot g) \circ \alpha)'(0) \\
&= ((f \circ \alpha) \cdot (g \circ \alpha))'(0) \\
&= ((f \circ \alpha)' \cdot (g \circ \alpha) + (f \circ \alpha) \cdot (g \circ \alpha)')(0) \\
&= v_{[\alpha]}(f) \cdot g(p) + f(p) \cdot v_{[\alpha]}(g).
\end{aligned}$$

Finally, we show that Φ_1 is linear. Take $[\alpha], [\beta] \in T_p^{\text{geom}} M$ and let $\lambda \in \mathbb{R}$. Suppose that $[\gamma] = \lambda[\alpha] + [\beta]$ for some $\gamma \in \mathcal{K}_p(M)$. So we have

$$v_{\lambda[\alpha] + [\beta]}(f) = (f \circ \gamma)'(0).$$

Choosing a chart (U, h) at p , recall that by definition of the linear structure on T^{geom} ,

$$(h \circ \gamma)'(0) = \lambda(h \circ \alpha)'(0) + (h \circ \beta)'(0).$$

A straightforward computation then gives us the linearity of Φ_1 :

$$\begin{aligned}
(f \circ \gamma)'(0) &= J(f \circ h^{-1} \circ h \circ \gamma)(0) \\
&= J(f \circ h^{-1})(h(p))J(h \circ \gamma)(0) \\
&= J(f \circ h^{-1})(h(p))(h \circ \gamma)'(0) \\
&= J(f \circ h^{-1})(h(p))(\lambda(h \circ \alpha)'(0) + (h \circ \beta)'(0)) \\
&= J(f \circ h^{-1})(h(p))(\lambda J(h \circ \alpha)(0) + J(h \circ \beta)(0)) \\
&= \lambda J(f \circ h^{-1})(h(p))J(h \circ \alpha)(0) + J(f \circ h^{-1})(h(p))J(h \circ \beta)(0) \\
&= \lambda J(f \circ h^{-1} \circ h \circ \alpha)(0) + J(f \circ h^{-1} \circ h \circ \beta)(0) \\
&= \lambda(f \circ \alpha)'(0) + (f \circ \beta)'(0) \\
&= \lambda v_{[\alpha]}(f) + v_{[\beta]}(f).
\end{aligned}$$

2. $T_p^{\text{alg}}(M) \longrightarrow T_p^{\text{phy}}(M)$.

Define $\Phi_2: T_p^{\text{alg}}(M) \longrightarrow T_p^{\text{phy}}(M)$ as follows:

$$\begin{aligned}
\Phi_2: T_p^{\text{alg}}(M) &\longrightarrow T_p^{\text{phy}}(M) \\
v &\longmapsto \bar{v}: (U, h) \mapsto (v(h_1), \dots, v(h_n))
\end{aligned}$$

where $v: \mathcal{E}_p(M) \rightarrow \mathbb{R}$ is a linear derivation (defined on the germs of differentiable functions on M at p), and h_i is the i -th component of h .

Linearity of Φ_2 is straightforward. We need to show that $\bar{v}$ behaves well under a change of coordinates. Let (V, k) be another chart at p . We want $\bar{v}(V, k) = D_{h(p)}(k \circ h^{-1})(\bar{v}(U, h))$. That is,

$$J(k \circ h^{-1})(h(p)) \begin{pmatrix} v(h_1) \\ \vdots \\ v(h_n) \end{pmatrix} = \begin{pmatrix} v(k_1) \\ \vdots \\ v(k_n) \end{pmatrix}.$$

Use $w = (w_1, \dots, w_n)$ to denote $k \circ h^{-1}$, and, for convenience, assume that $h(p) = k(p) = 0$. By Lemma 4.1, we can write

$$w_i = \sum_{j=1}^n x_j w_{i,j}$$

where $w_{i,j}(x) = \int_0^1 \frac{\partial w_i}{\partial x_j}(tx) dt$. Taking partial derivative on both sides of the above displayed equation with respect to some x_ℓ and evaluating at zero:

$$(1) \quad \frac{\partial w_i}{\partial x_\ell}(0) = \sum_{j=1}^n \frac{\partial x_j}{\partial x_\ell} \Big|_{x=0} w_{i,j}(0) + \sum_{j=1}^n x_j \Big|_{x=0} \frac{\partial w_{i,j}}{\partial x_\ell}(0) = w_{i,\ell}(0).$$

Note that

$$k(x) = (k \circ h^{-1} \circ h)(x) = (w \circ h)(x) = (w_1(h(x)), \dots, w_n(h(x))),$$

and, therefore,

$$k_i(x) = w_i(h(x)) = \sum_{j=1}^n h_j(x) w_{i,j}(h(x)).$$

Since v is a derivation (and recall that by assumption $h(p) = k(p) = 0$),

$$v(k_i) = \sum_{j=1}^n (v(h_j) w_{i,j}(h(p)) + h_j(p) v(w_{i,j} \circ h)) = \sum_{j=1}^n v(h_j) w_{i,j}(0).$$

Therefore, by Equation 1, we have

$$v(k_i) = \sum_{j=1}^n v(h_j) \frac{\partial w_i}{\partial x_j}(0).$$

Write this out in matrix form:

$$\begin{bmatrix} \vdots \\ \frac{\partial w_i}{\partial x_1}(0) & \dots & \frac{\partial w_i}{\partial x_n}(0) \\ \vdots \end{bmatrix} \begin{bmatrix} v(h_1) \\ \vdots \\ v(h_n) \end{bmatrix} = \begin{bmatrix} v(k_1) \\ \vdots \\ v(k_n) \end{bmatrix},$$

to see that this is exactly what we want.

3. $T_p^{\text{phy}}(M) \longrightarrow T_p^{\text{geom}}(M)$.

Given $v \in T_p^{\text{phy}}(M)$ and a chart (U, h) at p , define a curve $\alpha_v: (-\varepsilon, \varepsilon) \rightarrow M$ as follows: Pick a curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow h(U) \subseteq \mathbb{R}^n$ such that $\gamma(0) = h(p)$ and $\gamma'(0) = v(U, h)$. To be specific, say $\gamma(t) = h(p) + tv(U, h)$ for small enough ε . Then let $\alpha_v := h^{-1} \circ \gamma: (-\varepsilon, \varepsilon) \rightarrow M$. Define $\Phi_3: T_p^{\text{phy}}(M) \longrightarrow T_p^{\text{geom}}(M)$ to be the map that assigns $v \in T_p^{\text{phy}}(M)$ to $[\alpha_v]$.

The main thing to check is that the equivalence class of α_v is independent of the choice of chart. Let (V, k) be another chart at p and define another curve

$$\beta(t) = k^{-1}(k(p) + tv(V, k)).$$

Then $(k \circ \beta)'(0) = v(V, k)$. To see that $[\alpha_v] = [\beta]$, we calculate

$$\begin{aligned} (k \circ \alpha)'(0) &= J(k \circ \alpha)(0) \\ &= J(k \circ h^{-1} \circ h \circ \alpha)(0) \end{aligned}$$

$$\begin{aligned}
&= J(k \circ h^{-1})(h(p))J(h \circ \alpha)(0) \\
&= J(k \circ h^{-1})(h(p))v(U, h) \\
&= v(V, k) \\
&= (k \circ \beta)'(0),
\end{aligned}$$

where the penultimate step follows since v is a physically-defined tangent vector.

We summarize our work in Figure 9.

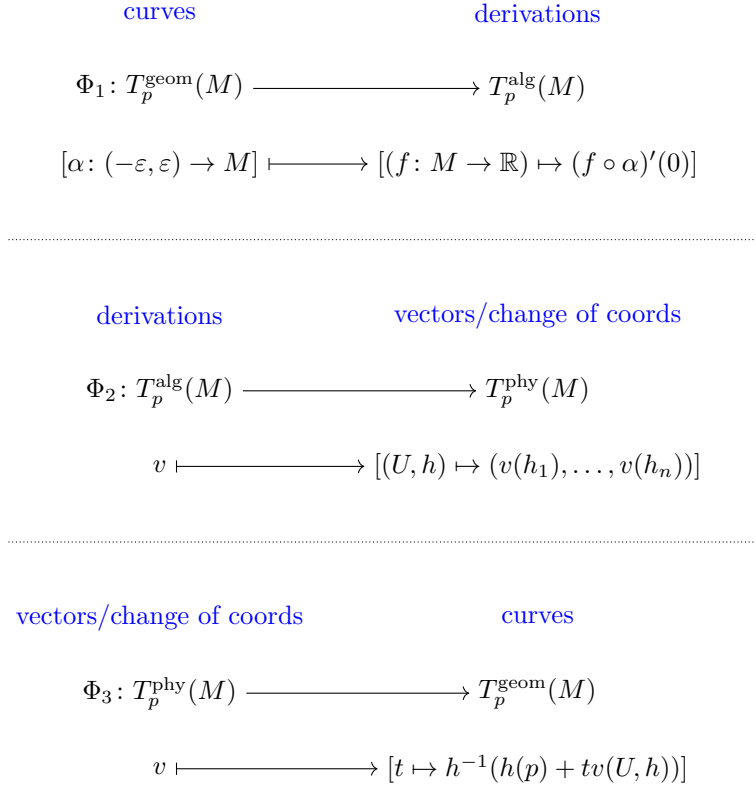


FIGURE 9. The linear isomorphisms Φ_1 , Φ_2 , and Φ_3 .

The last thing that needs to be shown is that $\Phi_3 \circ \Phi_2 \circ \Phi_1$ is the identity on $T_p^{\text{geom}}(M)$. Beside commutativity of the diagram in Proposition 4.11, this result will imply that all of the Φ_i are linear isomorphisms since we have already established that all three versions of tangent space are n -dimensional.

Given a curve $[\alpha] \in T_p^{\text{geom}}M$, the resulting curve after applying $\Phi_3 \circ \Phi_2 \circ \Phi_1$ is the equivalence class of the curve β , where

$$\beta(t) = h^{-1}(h(p) + t(h \circ \alpha)'(0))$$

(Note that the i -th component of $(\Phi_2 \circ \Phi_1)([\alpha])$ is exactly $(h_i \circ \alpha)'(0)$). Then, since h is a homeomorphism,

$$(h \circ \beta)(t) = h(p) + t(h \circ \alpha)'(0).$$

We can therefore conclude that $(h \circ \beta)'(0) = (h \circ \alpha)'(0)$, and $\alpha \sim \beta$ as desired. $\square$

4.3. Standard bases. We have now shown in precisely what sense the three spaces $T_p^{\text{geom}}M$, $T_p^{\text{alg}}M$, and $T_p^{\text{phy}}M$ are actually the same object. Thus, we are safe to talk about *the* tangent space to M at p , denoted T_pM , and use any of the three definitions to denote a tangent vector at p .

Here we define the *standard basis* for T_pM with respect to chart (U, h) at p , denoted

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

As an element of $T_p^{\text{geom}}(M)$, we define $(\frac{\partial}{\partial x_i})_p$ to be the equivalence class of curves represented by

$$t \mapsto h^{-1}(h(p) + te_i)$$

where e_i is the i -th standard basis vector of $\mathbb{R}^n$. As an element of $T_p^{\text{alg}}M$, i.e., as a derivation, for f a germ at p , we define

$$\left(\frac{\partial}{\partial x_i}\right)_p f := \frac{\partial}{\partial x_i}(f \circ h^{-1})(h(p)).$$

And finally, as an element of $T_p^{\text{phy}}M$, define $(\frac{\partial}{\partial x_i})_p := e_i$, the i -th standard basis vector of $\mathbb{R}^n$.

Our previous discuss of the three versions of tangent space have shown that these are bases, and one can check that they are compatible with our isomorphisms Φ_i .

4.4. The differential of a mapping of manifolds. Let $f: M \rightarrow N$ be a differentiable mapping of manifolds and let $p \in M$. Finally, we are ready to define the *differential* of f at p . It is a linear mapping $df_p: T_pM \rightarrow T_{f(p)}N$ with the following three descriptions compatible with the maps Φ_1, Φ_2, Φ_3 :

- **Geometric.**

$$\begin{aligned} d^{\text{geom}}f_p: T_p^{\text{geom}}M &\longrightarrow T_{f(p)}^{\text{geom}}N \\ [\alpha] &\longmapsto [f \circ \alpha] \end{aligned}$$

- **Algebraic.**

Precomposing by f assigns germs at p to germs at $f(p)$ (see Figure 10). So f induces an *algebra homomorphism*, $f^*: \mathcal{E}_{f(p)}(N) \longrightarrow \mathcal{E}_p(M)$ that sends $\phi \in \mathcal{E}_{f(p)}(N)$ to $\phi \circ f \in \mathcal{E}_p(M)$. Therefore, precomposing by f turns a derivation at p to a derivation at $f(p)$. We define the differential to be

$$\begin{aligned} d^{\text{alg}}f_p: T_p^{\text{alg}}M &\longrightarrow T_{f(p)}^{\text{alg}}N \\ v &\longmapsto v \circ f^* \end{aligned}$$

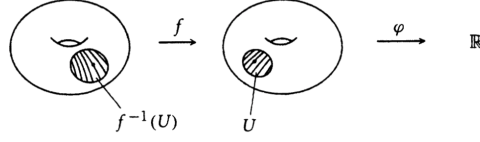


FIGURE 10. The germ of $\phi \circ f|_{f^{-1}(U)}$ at p is assigned to the germ of $\phi: U \rightarrow \mathbb{R}$ at $f(p)$.

Where $v \circ f^*$ is the derivation that sends a germ ϕ at $f(p)$ to the germ $v(\phi \circ f)$ at p (again, see Figure 10).

- **Physical.**

Given a chart (V, k) at $f(p)$, choose a chart (U, h) at p such that $f(U) \subseteq V$. For $v \in T_p^{\text{phy}} M$, define

$$(d^{\text{phy}} f_p(v))(V, k) := D_{h(p)}(k \circ f \circ h^{-1})v(U, h).$$

Thus, once local coordinates are taken, the differential is the ordinary derivative mapping given by the Jacobian matrix. See Figure 11.

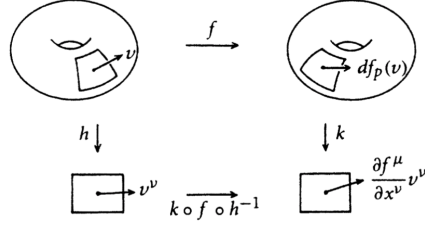


FIGURE 11. The differential in terms of physically-defined tangent spaces.

Example 4.12. Consider

$$\begin{aligned} f: \mathbb{P}^2 &\rightarrow \mathbb{P}^3 \\ (x, y, z) &\mapsto (x^3, y^3, z^3, xyz). \end{aligned}$$

Let $p = (1, s, t) \in U_x$. Then $f(p) = (1, s^3, t^3, st)$. Consider the standard open set $V_a = \{(a, b, c, d) \in \mathbb{P}^3 \mid a \neq 0\}$ with coordinate mapping $\phi_a(a, b, c, d) = (b/a, c/a, d/a)$. With respect to these charts, we have

$$\tilde{f}(u, v) = (\phi_a \circ f \circ \phi_x^{-1})(u, v) = (u^3, v^3, uv).$$

Its Jacobian matrix at $\phi_x(p) = (s, t)$ is

$$J\tilde{f}(s, t) = \begin{bmatrix} 3s^2 & 0 \\ 0 & 3t^2 \\ t & s \end{bmatrix}.$$

So given $v \in T_p^{\text{phy}} M$ that assigns the vector (v_1, v_2) to (U_x, ϕ_x) , we have that $df_p(v)$ assigns

$$J\tilde{f}(s, t) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

to (V_a, ϕ_a) .

Remark 4.13. The differential is functorial! The differential of the identity of M is the identity of $T_p M$:

$$d\text{id}_p = \text{id}_{T_p M}.$$

The differential also respects composition, i.e., the chain rule holds. For a composition $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ of differentiable maps, we have

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

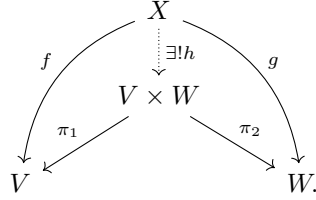
5. LINEAR ALGEBRA

Here we will summarize what we need to know about tensors. The presentation is mostly extracted from Frank Warner's, *Foundations of Differential Manifolds and Lie Groups*. In the following, all vector spaces are finite-dimensional and defined over an arbitrary field $\mathbb{k}$ unless otherwise specified.

5.1. Products and coproducts. Let V and W be vector spaces. The vector space *product* of V and W , is the Cartesian product $V \times W$ with linear structure

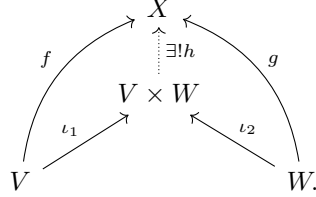
$$\lambda(v, w) + (v', w') := (\lambda v + v', w + w')$$

for all $\lambda \in \mathbb{k}$ and $(v, w), (v', w') \in V \times W$. It is the unique vector space (up to isomorphism) having the following *universal property*: Given a vector space X and linear mappings $f: X \rightarrow V$ and $g: X \rightarrow W$, there exists a linear mapping $h: X \rightarrow V \times W$ making the following diagram commute:



Here $\pi_1(v, w) = v$ and $\pi_2(v, w) = w$ are the first and second projection mappings, respectively. The mapping h is given by $h(x) = (f(x), g(x))$.

Similarly, define the *coproduct* of V and W , denoted $V \oplus W$, by $V \oplus W = V \times W$ with the same vector space structure. It has the universal property that given linear mappings $f: V \rightarrow X$ and $g: W \rightarrow X$ to a vector space X , there exists a unique linear mapping $h: V \oplus W \rightarrow X$ making the following diagram commute:



Here $\iota_1(v) = (v, 0)$ and $\iota_2(w) = (0, w)$ are the first and second inclusion mappings, respectively. The mapping h is determined by $h(v, w) = f(v) + g(w)$. Notice how the commutative diagram for coproducts is obtained from that for products by flipping the direction of the arrows.

We could also define the product $V_1 \times \cdots \times V_n$ and the coproduct $V_1 \oplus \cdots \oplus V_n$ for vector spaces $V_1, \dots, V_n$ by slightly extending the definition given above for the case $n = 2$. We will leave that to the reader along with the statement and verification of the corresponding universal properties.

It may seem peculiar to make a distinction between the product and coproduct here given that they are exactly the same vector spaces. The difference comes when we consider an infinite family $\{V_\alpha\}_{\alpha \in A}$ of vector spaces. In that case, we can define their product $\prod_{\alpha \in A} V_\alpha$ using the Cartesian product, just as above. However,

their coproduct $\coprod_{\alpha \in A} V_\alpha$ is the vector subspace of $\prod_{\alpha \in A} V_\alpha$ consisting of vectors for which all but a finite number of components are zero. In that way, the universal properties are satisfied. (To see why the definition of the coproduct must change in this case, note that for the coproduct of two vector spaces, the mapping h was given by $h(x) = f(x) + g(x)$. For an infinite family, the corresponding mapping would involve an infinite sum, and infinite sums of vectors are not defined in a general vector space.)

5.2. Tensor Products. Tensor products of vector spaces will allow us to think of multilinear objects (such as scalar products or determinants, and their generalizations) in terms of linear objects. We start with an informal description of the tensor product $U \otimes V \otimes W$ of three vector spaces U, V , and W . Its elements are linear combinations of expressions of the form $u \otimes v \otimes w$ where u, v , and w are elements in U, V , and W , respectively. We are not allowed to swap the vectors, i.e., $v \otimes u \otimes w \neq u \otimes v \otimes w$, in general. The defining property of the tensor is that it is, roughly speaking, linear with respect to each entry. Thus, for example, if $\alpha \in \mathbb{k}$, $u' \in U$ and $v' \in V$,

$$(\alpha u + u') \otimes v \otimes w = \alpha(u \otimes v \otimes w) + u' \otimes v \otimes w,$$

and

$$u \otimes (\alpha v + v') \otimes w = \alpha(u \otimes v \otimes w) + u \otimes v' \otimes w,$$

and similarly for the last component. As a last example, let $w' \in W$ and compute, using multilinearity:

$$\begin{aligned} (2u + u') \otimes v \otimes (4w - 3w') &= (2u) \otimes v \otimes (4w - 3w') + u' \otimes v \otimes (4w - 3w') \\ &= (2u) \otimes v \otimes (4w) + (2u) \otimes v \otimes (-3w') \\ &\quad + u' \otimes v \otimes (4w) + u' \otimes v \otimes (-3w') \\ &= 8u \otimes v \otimes w - 6u \otimes v \otimes w' \\ &\quad + 4u' \otimes v \otimes w - 3u' \otimes v \otimes w'. \end{aligned}$$

Example 5.1. Let e_1, e_2 be the standard basis vectors for $\mathbb{R}^2$, and let f_1, f_2, f_3 be the standard basis vectors for $\mathbb{R}^3$. Take $v = (2, 3) \in \mathbb{R}^2$ and $w = (3, 2, 1) \in \mathbb{R}^3$. Then we can write $v \otimes w \in \mathbb{R}^2 \otimes \mathbb{R}^3$ in terms of the $e_i \otimes f_j$:

$$\begin{aligned} v \otimes w &= (2, 3) \otimes (3, 2, 1) \\ &= (2e_1 + 3e_2) \otimes (3f_1 + 2f_2 + f_3) \\ &= 6e_1 \otimes f_1 + 4e_1 \otimes f_2 + 2e_1 \otimes f_3 + 9e_2 \otimes f_1 + 6e_2 \otimes f_2 + 3e_2 \otimes f_3. \end{aligned}$$

If you follow the above calculations, then you understand exactly the type of gadget we are looking for. We pause now for the formal construction (which is not as important as understanding the above calculation). We then describe the purpose of the tensor product by exhibiting its universal property.

5.2.1. *Construction of the tensor product.* To construct the tensor product $V \otimes W$ of vector spaces V and W , let $F(V, W)$ be the free vector space on the set of symbols $\{[v, w] \mid v \in V, w \in W\}$. Thus, these symbols form a basis for $F(V, W)$: an arbitrary element of $F(V, W)$ is a linear combination of these symbols and there is no relation among them. For instance, $[v, w]$, $[v', w]$ and $[v + v', w]$ are linearly independent if v , v' , and $v + v'$ are distinct.

We now mod out by a subspace of $F(V, W)$ in order to force the resulting equivalence classes of the $[v, w]$ to be “multilinear”. To that end, define T to be the subspace of $F(V, W)$ generated by the following vectors:

$$\begin{aligned} [v_1 + v_2, w] - [v_1, w] - [v_2, w] \\ [v, w_1 + w_2] - [v, w_1] - [v, w_2] \\ [\alpha v, w] - \alpha[v, w] \\ [v, \alpha w] - \alpha[v, w] \end{aligned}$$

for all $\alpha \in \mathbb{k}$, $v, v_1, v_2 \in V$, and $w, w_1, w_2 \in W$. Finally, define

$$V \otimes W := F(V, W)/T$$

and $v \otimes w := [v, w] \bmod T$ for each $v \in V$ and $w \in W$. (Modding out by T forces (the equivalence class of) each of the generators listed above to be 0, which gives just the multilinearity we want.) The tensor product of vector spaces $V_1, \dots, V_n$ is defined similarly.

Remark 5.2. Note that scalars can “float around” in tensors: for $\alpha \in \mathbb{k}$ and $u \otimes v \otimes w \in U \otimes V \otimes W$,

$$\alpha(u \otimes v \otimes w) = (\alpha u) \otimes v \otimes w = u \otimes (\alpha v) \otimes w = u \otimes v \otimes (\alpha w).$$

5.2.2. *Universal property of the tensor product.* Define

$$\begin{aligned} \iota: V \times W &\rightarrow V \otimes W \\ (v, w) &\mapsto v \otimes w, \end{aligned}$$

and note that ι is bilinear $\iota(\alpha v + v', w) = \alpha \iota(v, w) + \iota(v', w)$, and similarly for the second component). The tensor product $V \otimes W$ is characterized (up to isomorphism of vector spaces) by the following universal property: Given any bilinear mapping $f: V \times W \rightarrow U$ to a vector space U , there exists a unique linear mapping $h: V \otimes W \rightarrow U$ such that the following diagram commutes:

$$\begin{array}{ccc} & V \otimes W & \\ \iota \nearrow & & \searrow \exists! h \\ V \times W & \xrightarrow{f} & U. \end{array}$$

Thus, the tensor product allows us to represent a bilinear mapping with a linear mapping—each contains the same information. The proof of the universal property is left as an exercise.

More generally, there is a similar commutative diagram that relates a multilinear mapping $V_1 \times \dots \times V_n \rightarrow U$ with a linear mapping $V_1 \otimes \dots \otimes V_n \rightarrow U$:

$$\begin{array}{ccc}
& & V_1 \otimes \cdots \otimes V_\ell \\
& \nearrow \iota & \\
V_1 \times \cdots \times V_\ell & \xrightarrow{\text{multilinear}} & U.
\end{array}
\quad \begin{array}{c} \text{---} \\ \exists ! \text{linear} \end{array}$$

5.2.3. *Identities.* A typical use of the universal property of tensor is to define a linear mapping $V \otimes W \rightarrow U$ for some vector space. One could imagine defining a function by describing the image of an element $v \otimes w$ in terms of some rule involving v and w . The problem is that each $v \otimes w$ is really an equivalence class in $F(V, W)$. So there is the question of whether the mapping is well-defined. To get around that problem, one instead defines a bilinear mapping $f: V \times W \rightarrow U$. By the universal property of tensor products, there is then an induced linear mapping $\tilde{f}: V \otimes W \rightarrow U$ with the property that

$$\tilde{f}(v \otimes w) = f(v, w).$$

Also note that any linear mapping $f: V \otimes W \rightarrow U$ is determined by its values on tensors of the form $v \otimes w$. Not every element of $V \otimes W$ has the form $v \otimes w$ for some choices of v and w . However, elements of that form span $V \otimes W$.

The proof of the following proposition illustrates the principle of using the universal property to define mappings involving tensors.

Proposition 5.3. *Let U, V and W be vector spaces over $\mathbb{k}$.*

- (1) $V \otimes \mathbb{k} \approx V$.
- (2) $V \otimes W \approx W \otimes V$.
- (3) $(V \otimes W) \otimes U \approx V \otimes (W \otimes U) \approx V \oplus W \oplus U$.
- (4) $V \otimes (W \oplus U) \approx (V \otimes W) \oplus (V \otimes U)$.

Proof. We will prove the first two parts, the others being similar.

- (1) Define $f: V \rightarrow V \otimes \mathbb{k}$ by $f(v) = v \otimes 1$. It is straightforward to check that f is linear. To define the inverse, note that

$$\begin{aligned}
V \times \mathbb{k} &\rightarrow V \\
(v, \alpha) &\mapsto \alpha v
\end{aligned}$$

is bilinear. It, thus, induces a linear mapping $g: V \otimes \mathbb{k} \rightarrow V$ determined by $g(v \otimes \alpha) = \alpha v$. We then have

$$(f \circ g)(v \otimes \alpha) = f(\alpha v) = (\alpha v) \otimes 1 = v \otimes \alpha.$$

and

$$(g \circ f)(v) = g(f(v)) = g(v \otimes 1) = 1 \cdot v = v.$$

- (2) The bilinear mapping

$$\begin{aligned}
V \times W &\rightarrow W \otimes V \\
(v, w) &\mapsto w \otimes v
\end{aligned}$$

induces a linear mapping $f: V \otimes W \rightarrow W \otimes V$ with the property that $f(v \otimes w) = w \otimes v$. By a similar argument, there is a linear mapping $g: W \otimes V \rightarrow V \otimes W$ with the property that $g(w \otimes v) = v \otimes w$. It is then easy to check that f and g are inverses.

□

Proposition 5.4. *Let V and W be vector spaces with bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$, respectively. Then $V \otimes W$ has basis $\{v_i \otimes w_j\}_{i,j}$. In particular,*

$$\dim(V \otimes W) = \dim(V) \dim(W).$$

Proof. Using the ideas presented above, we can define a sequence of isomorphisms

$$V \otimes W \approx V \otimes (\oplus_{j=1}^n \mathbb{k}) \approx \oplus_{j=1}^n (V \otimes \mathbb{k}) \approx \oplus_{j=1}^n V \approx \oplus_{j=1}^n \oplus_{i=1}^m \mathbb{k} \approx \mathbb{k}^{mn}$$

sending the $v_i \otimes w_j$ to the standard basis vectors for $\mathbb{k}^{mn}$. □

5.3. Symmetric and exterior products. If $u \otimes v \in V \otimes V$ and $u \neq v$, then it is typically the case that $u \otimes v \neq v \otimes u$. (For an exception, we have $u \otimes (\alpha u) = (\alpha u) \otimes u$ for any scalar α .) However, there are many situations in which such a commutativity property would be desirable. Thus, we are led to the notion of symmetric tensors. For $\ell \in \mathbb{N}$, define

$$V^{\otimes \ell} := \underbrace{V \otimes \cdots \otimes V}_{\ell}.$$

We take $V^{\otimes 0} := \mathbb{k}$. Consider the subspace T of $V^{\otimes \ell}$ generated by elements of the form

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_\ell - v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_\ell$$

formed by swapping two components of the tensor. Then the ℓ -th symmetric product of V is quotient vector space

$$\text{Sym}^\ell V = V^{\otimes \ell} / T.$$

The element $v_1 \otimes \cdots \otimes v_\ell \in \text{Sym}^\ell V$ is denoted $v_1 \cdots v_\ell$, and we are allowed to commute the v_i without changing the element. Elements of $\text{Sym}^\ell V$ are called *symmetric tensors*. The elements of $\text{Sym}^\ell V$ behave just like usual tensors, except we are now allowed to swap components.

A multilinear function $f: V^{\times \ell} \rightarrow W$ is *symmetric* if $f(v_1, \dots, v_i, \dots, v_j, \dots, v_\ell) = f(v_1, \dots, v_j, \dots, v_i, \dots, v_\ell)$ for all i, j . The universal property of symmetric tensors is that given a multilinear symmetric mapping $V^{\times \ell} \rightarrow W$ from the ℓ -fold Cartesian product of V with itself to W , there exists a unique linear mapping $\text{Sym}^\ell V \rightarrow W$ making the following diagram commute:

$$\begin{array}{ccc} & \text{Sym}^\ell V & \\ \nearrow & & \searrow \exists! \\ V^{\times \ell} & \xrightarrow{\text{multilinear, symmetric}} & W. \end{array}$$

The mapping $V^{\times \ell} \rightarrow \text{Sym}^\ell V$ is the natural one determined by $(v_1, \dots, v_\ell) \mapsto v_1 \cdots v_\ell$.

Example 5.5. The ordinary dot product:

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

$$(v, w) \mapsto \langle v, w \rangle = \sum_{i=1}^n v_i w_i$$

is bilinear. Hence, there is a unique induced mapping $\text{Sym}^2 \mathbb{R}^n \rightarrow \mathbb{R}$ making the following diagram commute.

$$\begin{array}{ccc} & \text{Sym}^2 \mathbb{R}^n & \\ \nearrow & & \searrow \\ \mathbb{R}^n \times \mathbb{R}^n & \xrightarrow{\quad} & \mathbb{R}. \end{array}$$

It is determined by $(v, w) \mapsto \langle v, w \rangle$ for all $v, w \in \mathbb{R}^n$.

Proposition 5.6. *If V has dimension n , then*

$$\dim \text{Sym}^\ell V = \binom{n + \ell - 1}{\ell}.$$

Proof. Exercise. If V has basis $x_1, \dots, x_n$, show that $\text{Sym}^\ell V$ has basis

$$\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \geq 0 \text{ and } \sum_i a_i = \ell\},$$

i.e., the basis consists of all monomials of degree ℓ in $x_1, \dots, x_n$. The result then follows from the usual stars-and-bars argument. $\square$

We also define

$$\text{Sym } V := \bigoplus_{\ell \geq 0} \text{Sym}^\ell V$$

where $\text{Sym}^0 V := \mathbb{k}$. One may use the universal property for symmetric products to show there is a well-defined multiplication mapping

$$\begin{aligned} \text{Sym}^a V \times \text{Sym}^b V &\rightarrow \text{Sym}^{a+b} V \\ (\alpha, \beta) &\mapsto \alpha\beta. \end{aligned}$$

This multiplication turns $\text{Sym } V$ into a $\mathbb{k}$ -algebra (i.e., a vector space over $\mathbb{k}$ which is also a ring (i.e., has a nice multiplication operation)). In fact, if V has basis $x_1, \dots, x_n$, then $\text{Sym } V$ is isomorphic to the polynomial ring in $x_1, \dots, x_n$ with coefficients in $\mathbb{k}$ (but see the section on duality, below, to get a better formulation).

In the same way we defined symmetric products by forcing tensors to commute, we define exterior products by forcing tensors to anti-commute.

The ℓ -th exterior product of V , denoted $\Lambda^\ell V$, to be the quotient vector space

$$\Lambda^\ell V := V^{\otimes \ell} / T,$$

where $T = \text{Span}_{\mathbb{k}}\{v_1 \otimes \cdots \otimes v_\ell \mid v_i = v_j \text{ for some } i \neq j\}$. The equivalence class of $v_1 \otimes \cdots \otimes v_\ell$ is denoted $v_1 \wedge \cdots \wedge v_\ell$. Elements of $\Lambda^\ell V$ are called *alternating tensors*.

Proposition 5.7. *In $\Lambda^\ell V$, we have, swapping v_i and v_j for any $i \neq j$,*

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_\ell = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_\ell.$$

Proof. Through direct computation:

$$\begin{aligned} 0 &= v_1 \wedge \cdots \wedge (v_i + v_j) \wedge \cdots \wedge (v_i + v_j) \wedge \cdots \wedge v_\ell \\ &= 0 + v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_\ell + v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_\ell + 0. \end{aligned}$$

□

Remark 5.8. If $\text{char } \mathbb{k} \neq 2$, i.e., if $1 + 1 \neq 0$ in $\mathbb{k}$, then we can use the equation displayed in the above proposition to define T .

A multilinear map $f: V^{\times \ell} \rightarrow W$ is *alternating* if $f(v_1, \dots, v_\ell) = 0$ if $v_i = v_j$ for some $i \neq j$. There is a canonical multilinear alternating mapping

$$\begin{aligned} \iota: V^{\times \ell} &\rightarrow \Lambda^\ell V \\ (v_1, \dots, v_\ell) &\mapsto v_1 \wedge \cdots \wedge v_\ell. \end{aligned}$$

The ℓ -th exterior product is characterized by the following universal property: given any vector space W and multilinear alternating mapping $V^{\times \ell} \rightarrow W$, there is a unique linear map $\Lambda^\ell V \rightarrow W$ making the following diagram commute:

$$\begin{array}{ccc} & \Lambda^\ell V & \\ \iota \nearrow & & \searrow \exists! \\ V^{\times \ell} & \xrightarrow{\text{multilinear, alternating}} & W. \end{array}$$

Example 5.9. Let $\mathbb{k} = \mathbb{R}$. The cross product $\times: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bilinear and alternating. So there is a unique map $\times: \Lambda^2 \mathbb{R}^n \rightarrow \mathbb{R}^n$ that sends $u \wedge v$ to $u \times v$ for any $u, v \in \mathbb{R}^n$.

Proposition 5.10. *If V has dimension n , then*

$$\dim \Lambda^\ell V = \binom{n}{\ell}.$$

Proof. Exercise. Show that if $e_1, \dots, e_n$ is a basis for V , then $\{e_{i_1} \wedge \cdots \wedge e_{i_\ell} \mid i_1 < \cdots < i_\ell\}$ is a basis for $\Lambda^\ell V$. □

Exercise 5.11. Recall that the *determinant* $\det$ of a square matrix over $\mathbb{k}$ is the unique multilinear alternating function of its rows that sends the identity matrix to 1. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{k}^n$ and let $v_1, \dots, v_n \in \mathbb{k}^n$. Show that

$$v_1 \wedge \cdots \wedge v_n = \det(v_1, \dots, v_n) e_1 \wedge \cdots \wedge e_n.$$

Exercise 5.12. Show that $v_1, \dots, v_\ell \in V$ are linearly dependent if and only if $v_1 \wedge \cdots \wedge v_\ell = 0$.

Remark 5.13. If $v_1, \dots, v_\ell$ are linearly independent, it is tempting to identify the one-dimensional space spanned by $v_1 \wedge \cdots \wedge v_\ell \in \Lambda^\ell V$ with the linear space spanned by $v_1, \dots, v_\ell$ in V . We will talk about this more when we talk about Grassmannians later in these notes.

Define

$$\Lambda^\bullet V := \bigoplus_{\ell \geq 0} \Lambda^\ell V$$

where $\Lambda^0 V := \mathbb{k}$. Using the universal property of exterior products, we can define a multiplication on alternating tensors:

$$\begin{aligned} \Lambda^r V \times \Lambda^s V &\rightarrow \Lambda^{r+s} V \\ (\lambda, \mu) &\mapsto \lambda \wedge \mu. \end{aligned}$$

The vector space $\Lambda^\bullet V$ with this multiplication is called the *Grassmann algebra* on V . Note that for $\lambda \in \Lambda^r V$ and $\mu \in \Lambda^s V$ we have

$$\lambda \wedge \mu = (-1)^{rs} \mu \wedge \lambda.$$

5.4. Dual spaces. Let $\text{hom}(V, W)$ denote the linear space of all linear mappings $V \rightarrow W$. If $f, g \in \text{hom}(V, W)$ and $\lambda \in \mathbb{k}$, then $\lambda f + g$ is defined by

$$(\lambda f + g)(v) := \lambda f(v) + g(v).$$

The *dual* of the vector space V is

$$V^* := \text{hom}(V, \mathbb{k}).$$

Exercise 5.14. If $v_1, \dots, v_n$ is a basis for V , define $v_i^* \in V^*$ for each i by

$$v_i^*(v_j) := \delta(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\{v_1^*, \dots, v_n^*\}$ is a basis for V^* . It is called the *dual basis* to $\{v_1, \dots, v_n\}$. Note that this exercise shows that V and V^* are isomorphic (if V is finite-dimensional). However, the isomorphism depends on a choice of basis.

A linear mapping $f: V \rightarrow W$ induces a dual linear mapping

$$\begin{aligned} f^*: W^* &\rightarrow V^* \\ \phi &\mapsto \phi \circ f \end{aligned}$$

as pictured below;

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ & & \searrow \phi \\ & & \mathbb{k}. \end{array}$$

The reader should check that dualization is functorial: $\text{id}_V: V \rightarrow V$ induces $\text{id}_{V^*}: V^* \rightarrow V^*$, and commutative diagrams are preserved:

$$\begin{array}{ccc} V & \xrightarrow{\quad} & W \\ & \searrow & \swarrow \\ & U & \end{array} \quad \dots \rightarrow \quad \begin{array}{ccc} V^* & \xleftarrow{\quad} & W^* \\ & \swarrow & \searrow \\ & U^* & \end{array}.$$

Proposition 5.15. *We have two isomorphisms:*

(1)

$$\begin{aligned}\Lambda^\ell V^* &\rightarrow (\Lambda^\ell V)^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell &\mapsto [v_1 \wedge \cdots \wedge v_\ell \mapsto \det(\phi_i(v_j))]\end{aligned}$$

(2) and

$$\begin{aligned}\text{Sym}^\ell V^* &\rightarrow (\text{Sym}^\ell V)^* \\ \phi_1 \cdots \phi_\ell &\mapsto [v_1 \cdots v_\ell \mapsto \sum_{\sigma \in \mathfrak{S}_\ell} \prod_{i=1}^{\ell} \phi_{\sigma(i)}(v_i)],\end{aligned}$$

where $\mathfrak{S}_\ell$ is the symmetric group of permutations of $[\ell] := \{1, \dots, \ell\}$.

Sketch of proof of (1). The mapping in (1) exists from the universal property of alternating tensors and the fact that the determinant is alternating and multilinear as a function of its rows.

We define the inverse mapping $(\Lambda^\ell V)^* \rightarrow \Lambda^\ell V^*$ in terms of a basis. Fix a basis $e_1, \dots, e_n$ for V , and for each integer vector μ such that

$$1 \leq \mu_1 < \cdots < \mu_\ell \leq n,$$

define

$$e_\mu := e_{\mu_1} \wedge \cdots \wedge e_{\mu_\ell}.$$

These $\binom{n}{\ell}$ vectors form a basis for $\Lambda^\ell V$. Letting $e_1^*, \dots, e_\ell^*$ be the corresponding dual basis for V^* , we get the corresponding basis of vectors

$$e_\mu^* := e_{\mu_1}^* \wedge \cdots \wedge e_{\mu_\ell}^*.$$

for $\Lambda^\ell(V^*)$. Define

$$\begin{aligned}g: (\Lambda^\ell V)^* &\longrightarrow \Lambda^\ell V^* \\ \omega &\longmapsto \sum_{\mu: 1 \leq \mu_1 < \cdots < \mu_\ell \leq n} \omega(e_\mu) e_\mu^*,\end{aligned}$$

where $e_\mu^* = e_{\mu_1}^* \wedge \cdots \wedge e_{\mu_\ell}^*$. The reader may check that g is well-defined and inverse to the mapping in (1). $\square$

5.4.1. Forms. A *multilinear ℓ -form* on V is a multilinear mapping $V^{\times \ell} \rightarrow \mathbb{k}$. By the universal property of tensor products, each corresponds to a mapping $V^{\otimes \ell} \rightarrow \mathbb{k}$, and hence to an element of $(V^{\otimes \ell})^*$. Thus, there is a linear isomorphism between the vector space of multilinear ℓ -forms on V and the vector space $(V^{\otimes \ell})^* \approx (V^*)^{\otimes \ell}$. A *symmetric ℓ -form* on V is a multilinear symmetric mapping $V^{\times \ell} \rightarrow \mathbb{k}$. By the universal property of symmetric tensors this mapping is identified with a mapping $\text{Sym}^\ell V \rightarrow \mathbb{k}$, i.e., to an element of $(\text{Sym}^\ell V)^*$. Thus, we get an isomorphism between the linear space of symmetric ℓ -forms and $(\text{Sym}^\ell(V))^* \approx \text{Sym}^\ell V^*$. An *alternating ℓ -form* on V is a multilinear alternating mapping $V^{\times \ell} \rightarrow \mathbb{k}$. Since it will be useful later, we will let $\text{Alt}^\ell V$ denote the linear space of alternating ℓ -forms on V . Arguing as above, the universal property of alternating tensors gives an isomorphism

$$\text{Alt}^\ell V \approx (\Lambda^\ell V)^* \approx \Lambda^\ell V^*.$$

Example 5.16. The space $\text{Sym}^2 V^*$ is the linear space of symmetric bilinear forms on V . For instance, any inner product $\langle \cdot, \cdot \rangle$ on V can be thought of as an element of $\text{Sym}^2 V^*$. (Recall that an *inner product* is a nondegenerate bilinear form. *Nondegenerate* means that if $v \in V$ and $\langle v, w \rangle = 0$ for all $w \in V$, then $v = 0$.)

Recall that the wedge product defined for the exterior algebra $\Lambda^\bullet V^*$ is given by

$$\begin{aligned} \wedge: \Lambda^r V^* \times \Lambda^s V^* &\longrightarrow \Lambda^{r+s} V^* \\ (\omega, \eta) &\mapsto \omega \wedge \eta. \end{aligned}$$

The product is bilinear, associative, and anti-commutative.

Exercise 5.17. Show that taking the exterior algebra of the dual space defines a contravariant functor $\Lambda^\bullet: V \mapsto \Lambda^\bullet V^*$ from the category of finite-dimensional $\mathbb{k}$ -vector spaces to the category of graded anti-commutative $\mathbb{k}$ -algebras. That is, show that (1) for $f: V \rightarrow W$ and $\omega, \eta \in \Lambda^\bullet W^*$, $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$, (2) for the composition $U \xrightarrow{f} V \xrightarrow{g} W$, we have $(g \circ f)^* = f^* \circ g^*: \Lambda^\bullet W^* \rightarrow \Lambda^\bullet V^* \rightarrow \Lambda^\bullet U^*$, and (3) id_V^* is the identity map on $\Lambda^\bullet V^*$.

5.4.2. *Pullbacks.* A linear map $L: V \rightarrow W$ induces, for each $\ell \geq 0$, a pullback mapping

$$\begin{aligned} L^*: (\Lambda^\ell W)^* &\rightarrow (\Lambda^\ell V)^* \\ \omega &\mapsto [v_1 \wedge \cdots \wedge v_\ell \mapsto \omega(Lv_1 \wedge \cdots \wedge Lv_\ell)], \end{aligned}$$

or equivalently (using the isomorphisms established above,

$$\begin{aligned} L^*: \Lambda^\ell W^* &\rightarrow \Lambda^\ell V^* &&\rightarrow (\Lambda^\ell V)^* \\ \phi_1 \wedge \cdots \wedge \phi_\ell &\mapsto L^*\phi_1 \wedge \cdots \wedge L^*\phi_\ell &\mapsto [v_1 \wedge \cdots \wedge v_\ell \mapsto \det((\phi_i \circ L)(v_j))]. \end{aligned}$$

6. VECTOR BUNDLES

Recall that in section 4, we learned three equivalent ways to define the tangent space $T_p M$ of a manifold M at a point $p \in M$. We are also interested in its dual space $T_p^* M$, called the *cotangent space*, as well as other related vector spaces $\Lambda^k T_p M$, $\Lambda^k T_p^* M$, $\text{Sym}^k T_p M$, $T_p^* M^{\otimes k}$, etc.

For each p , we can choose a chart U at p and look at all the tangent spaces at points in U . Properties of these tangent spaces give local properties of M . But to study the global properties, we need to look at all points in M and form a *vector bundle*. That is, to each point $p \in M$, we attach a vector space such that these vector spaces behave well under change of charts. Before studying specific bundles, we briefly introduce the general theory of vector bundles.

Definition 6.1. Let M be an n -dimensional manifold, E an $(n + r)$ -dimensional manifold, and $\pi: E \rightarrow M$ a smooth surjection. We say $\pi: E \rightarrow M$ is a *vector bundle of rank r* over M if

- (1) For every $p \in M$, the *fiber over p* , $E_p := \pi^{-1}(p)$ is a real vector space of dimension r ;
- (2) Every point $p \in M$ has an open neighborhood U such that there is a *fiber-preserving* diffeomorphism $\phi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^r$, meaning it makes the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathbb{R}^r \\ & \searrow \pi \quad \swarrow \pi_1 & \\ & U & \end{array}$$

where π_1 is the usual projection map onto the first coordinate. Furthermore, ϕ_U restricts to a linear isomorphism $E_p \rightarrow \{p\} \times \mathbb{R}^r$ on each fiber.

We call E the *total space*, M the *base space*, and the map ϕ_U a *trivialization of $\pi^{-1}(U)$* . A *line bundle* is a vector bundle of rank 1.

Example 6.2 (Product Bundle). Let V be a vector space of dimension r . Then the projection $\pi: M \times V \rightarrow M$ is a vector bundle of rank r . For example, the cylinder $S^1 \times \mathbb{R}$ together with the projection $\pi: S^1 \times \mathbb{R} \rightarrow S^1$ is a product bundle over S^1 .

Definition 6.3. Let $\pi_E: E \rightarrow M$ and $\pi_F: F \rightarrow N$ be vector bundles. A *bundle map* from E to F is a pair of differentiable maps $(\phi: E \rightarrow F, f: M \rightarrow N)$ such that

- (1) The following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ \pi_E \downarrow & & \downarrow \pi_F \\ M & \xrightarrow{f} & N \end{array}$$

- (2) ϕ restricts to a linear map $\phi_p: E_p \rightarrow F_{f(p)}$ for each $p \in M$.

Abusing language, we usually call $\phi: E \rightarrow F$ alone the bundle map.

If $M = N$, then we call the pair $(\phi: E \rightarrow F, \text{id}_M)$ a bundle map *over* M . Also, ϕ is a *bundle isomorphism over* M if there is another bundle map $\psi: F \rightarrow E$ over M such that $\phi \circ \psi = \text{id}_F$ and $\psi \circ \phi = \text{id}_E$.

Definition 6.4. A vector bundle $\pi: E \rightarrow M$ of rank r is *trivial* if it is isomorphic to the product bundle $M \times \mathbb{R}^r \rightarrow M$.

Example 6.5 (Möbius Strip). The open Möbius strip is the quotient of $[0, 1] \times \mathbb{R}$ by the identification $(0, t) \sim (1, -t)$. It gives a line bundle over S^1 that is not isomorphic to the cylinder. See Exercise 6.9.

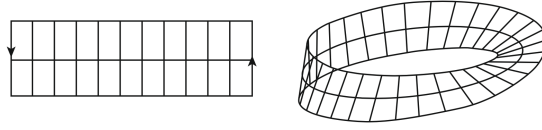


FIGURE 12. A Möbius strip.

Definition 6.6. Let $\pi: E \rightarrow M$ be a vector bundle and $U \subseteq M$ be open. A *section* of E over U is a smooth function $s: U \rightarrow E$ such that $\pi \circ s = \text{id}_U$. That is, for each $p \in U$, the section s picks out one element of the fiber E_p . We use $\Gamma(U, E)$ to denote the collection of all sections of E over U . If $U = M$, we also write $\Gamma(E)$ instead of $\Gamma(M, E)$ and call elements of $\Gamma(E)$ the *global sections*.

Remark 6.7. First note that $\Gamma(U, E)$ is a vector space over $\mathbb{R}$. Also note that there is an action on $\Gamma(U, E)$ by smooth functions on U : for a section $s \in \Gamma(U, E)$, a smooth function $f: U \rightarrow \mathbb{R}$, and a point $p \in U$, we define $(fs)(p) := f(p)s(p) \in E_p$. Thus, $\Gamma(U, E)$ is a $\mathcal{C}^\infty(U)$ -module.

Example 6.8 (Sections of a product line bundle). A section s of the product line bundle $M \times \mathbb{R}^k \rightarrow M$ is a map $s(p) = (p, f(p))$ for each $p \in M$ (see Figure 13). So

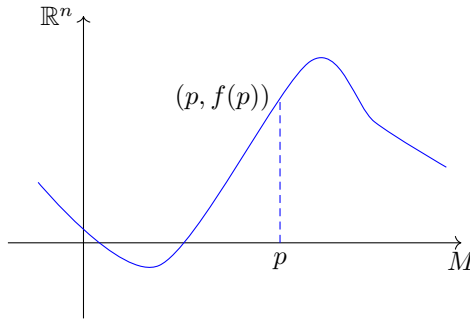


FIGURE 13. A section of the product bundle $M \times \mathbb{R}^k$

there is a bijection

$$\{\text{sections of } M \times \mathbb{R}^k \rightarrow M\} \leftrightarrow \{\text{smooth functions } f: M \rightarrow \mathbb{R}^k\}.$$

Exercise 6.9. Show that the open Möbius strip in Example 6.5 as a line bundle over S^1 is not trivial. (Hint: if it were, it would have a non-vanishing global section, e.g., $s(p) = 1$ for all $p \in S^1$. What is wrong with that?)

Remark 6.10. Given a global section $s \in \Gamma(E)$ and an open set $U \subseteq M$, we can always get a section $s|_U$ over U by restricting the domain of s . On the other hand, if $s \in \Gamma(U, E)$ is a section over U , then for every $p \in U$, we can find a global section $\bar{s}$ that agrees with s over some neighborhood of p (presumably contained in U) by multiplying s with a *bump function*.

Definition 6.11. A *frame* for a vector bundle E of rank r over an open set $U \subseteq M$ is a collection of sections $e_1, \dots, e_r$ of E over U such that at each point $p \in U$, the elements $e_1(p), \dots, e_r(p)$ form a basis for the fiber E_p .

Proposition 6.12. A vector bundle $\pi: E \rightarrow M$ of rank r is trivial if and only if it has a frame over M .

Proof. ($\Rightarrow$) Suppose that $\pi: E \rightarrow M$ is trivial with a bundle isomorphism $\phi: E \rightarrow M \times \mathbb{R}^r$. Let $u_1, \dots, u_r$ be the standard basis for $\mathbb{R}^r$. Then for every $p \in M$, the elements $(p, u_1), \dots, (p, u_r)$ form a basis for $\{p\} \times \mathbb{R}^r$. So the corresponding sections e_i over M defined by $e_i(p) := \phi^{-1}(p, u_i)$ form a basis for E_p .

($\Leftarrow$) Suppose that $e_1, \dots, e_r \in \Gamma(E)$ is a frame over M . Then for any $p \in M$ and $e \in E_p$, we have $e = \sum_{i=1}^r a_i e_i(p)$ for some $a_i \in \mathbb{R}$. Now define

$$\begin{aligned} \phi: E &\longrightarrow M \times \mathbb{R}^r \\ e &\longmapsto (p, a_1, \dots, a_r). \end{aligned}$$

This is a bundle map with inverse

$$\begin{aligned} \psi: M \times \mathbb{R}^r &\longrightarrow E \\ (p, a_1, \dots, a_r) &\longmapsto \sum_{i=1}^r a_i e_i(p). \end{aligned}$$

□

7. TANGENT BUNDLE

Definition 7.1. The *tangent bundle* of M , denoted TM , is the disjoint union of tangent spaces:

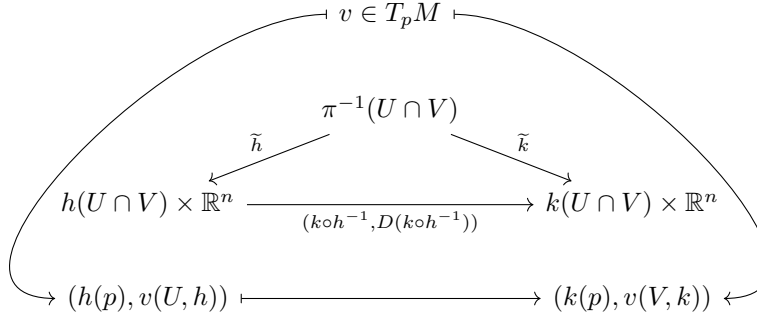
$$TM := \bigsqcup_{p \in M} T_p M,$$

together with a projection $\pi: TM \rightarrow M$ defined by $\pi(v) = p$ if $v \in T_p M$. We often denote an element of TM as (p, v) , meaning $v \in T_p M$.

The manifold structure on TM induced by the structure on M . For a chart (U, h) on M , we can form a chart $(\pi^{-1}(U), \tilde{h})$ on TM with $\tilde{h}$ defined by

$$\begin{aligned} \tilde{h}: \pi^{-1}(U) &\longrightarrow h(U) \times \mathbb{R}^n && \subseteq \mathbb{R}^n \times \mathbb{R}^n \\ v \in T_p M &\longmapsto (h(p), v(U, h)). \end{aligned}$$

Let (U, h) and (V, k) be charts at $p \in M$. The transition function of the corresponding charts in TM is given by $(k \circ h^{-1}, D(k \circ h^{-1}))$.

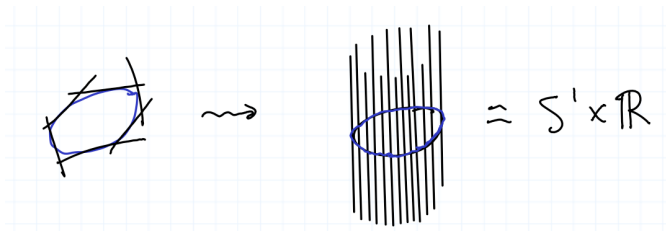


The differentiable structure on TM is chosen to be the maximal atlas containing these charts. To put a topology on TM , we say that $A \subseteq TM$ is open if $\tilde{h}(\pi^{-1}(U) \cap A)$ is open for each chart (U, h) . Then TM being Hausdorff and second countable follows. Note that with this structure, the mapping π is differentiable.

Exercise 7.2. Check that TM is Hausdorff and second countable. Conclude that TM is a differentiable manifold and that $\pi: TM \rightarrow M$ is a vector bundle over M of rank n (recall the vector space structure of $T_p M$).

Example 7.3. Let $M = \mathbb{R}^n$. Fix the chart $(\mathbb{R}^n, \text{id})$ for M . This induces the map $T_p M \cong \mathbb{R}^n$ for all $p \in M$. Then $TM = \pi^{-1}(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n$. We can think of $T\mathbb{R}^n$ as the result of attaching a copy of $\mathbb{R}^n$ to each point in $\mathbb{R}^n$.

Exercise 7.4. Consider the circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. The following picture leads one to think that TS^1 is trivial:



Prove this fact. [Hint: see Proposition 6.12, parametrize S^1 , and use this parametrization to define a derivation $v(p)$ smoothly varying with p and never equal to the zero derivation.]

Definition 7.5. A global section $s: M \rightarrow TM$ of the tangent bundle TM is called a *vector field*.

Remark 7.6. It is not true that the tangent bundle is always trivial. For example, consider $M = S^2$. The “hairy ball” theorem says that there cannot be non-vanishing continuous vector fields on M . That is, if $s \in \Gamma(TS^2)$ is a global section, then there must be some $p \in M$ such that $s(p) = 0$. It follows that TS^2 , unlike TS^1 , is non-trivial.

Remark 7.7. Let $s: M \rightarrow TM$ be a vector field. The *zero locus* of s is the set $\{p \in M \mid s(p) = 0\}$. Note that this is well-defined since $s(p) = 0$ is not dependent on the chart.

Let $f: M \rightarrow N$ be a differentiable map of manifolds. Recall again in section 4 that we defined $df_p: T_pM \rightarrow T_{f(p)}N$, the differential of f at a point $p \in M$. Now we use this information to induce a bundle map $f_*: TM \rightarrow TN$ of tangent bundles and define df , the *differential* of f to be this induced map.

Take $p \in M$ and a chart (U, h) at p . Let (V, k) be a chart at $f(p)$ such that $f(U) \subseteq V$. Let $(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_m})_p$ and $(\frac{\partial}{\partial y_1})_{f(p)}, \dots, (\frac{\partial}{\partial y_n})_{f(p)}$ denote the bases for T_pM and $T_{f(p)}N$ correspondingly. Recall that for df_p , we have the following diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{df_p} & T_{f(p)}N \\ \downarrow & \searrow \scriptstyle \text{“} \frac{\partial f_i}{\partial x_j}(p) \text{”} & \downarrow \\ \mathbb{R}^m & \xrightarrow{J(k \circ f \circ h^{-1})(h(p))} & \mathbb{R}^n. \end{array}$$

To define f_* on TM , we first define f_* locally on the chart $\pi^{-1}(U)$:

$$\begin{array}{ccc} \pi_M^{-1}(U) & \xrightarrow{f_*} & \pi_N^{-1}(V) \\ \cong \downarrow & & \downarrow \cong \\ h(U) \times \mathbb{R}^m & \xrightarrow{(k \circ f \circ h^{-1}, df_p)} & k(V) \times \mathbb{R}^n. \end{array}$$

Then we can “glue” the pieces together since f_* commutes with the transition maps: Take two pairs of charts (U_i, h_i) and (V_i, k_i) with $f(U_i) \subseteq V_i$ for $i = 1, 2$. Let $\tilde{f}_i$ denote the composition $k_i \circ f \circ h_i^{-1}$ and observe that the following diagram commutes.

$$\begin{array}{ccc} h_1(U_1 \cap U_2) \times \mathbb{R}^m & \xrightarrow{(\tilde{f}_1, J\tilde{f}_1, (h(p)))} & k_1(V_1 \cap V_2) \times \mathbb{R}^n \\ h_2 \circ h_1^{-1} \downarrow & & \downarrow k_2 \circ k_1^{-1} \\ h_2(U_1 \cap U_2) \times \mathbb{R}^m & \xrightarrow{(\tilde{f}_2, J\tilde{f}_2, (h(p)))} & k_2(V_1 \cap V_2) \times \mathbb{R}^n. \end{array}$$

Thus, these local mappings glue to define the differential:

$$(2) \quad \begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N. \end{array}$$

Example 7.8. Let $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$, and consider the function

$$\begin{aligned} f: M &\rightarrow N \\ (x, y) &\mapsto (x, y, x^2 - y^2). \end{aligned}$$

Choosing the charts $(\mathbb{R}^2, \text{id}_{\mathbb{R}^2})$ and $(\mathbb{R}^3, \text{id}_{\mathbb{R}^3})$, the commutative diagram (2) becomes

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{df} & \mathbb{R}^3 \times \mathbb{R}^3 \\ \downarrow & & \downarrow \\ \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^3. \end{array}$$

with

$$\begin{aligned} df: \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \\ ((x, y), (u, v)) &\mapsto ((x, y, x^2 - y^2), (u, v, 2xu - 2yv)) \end{aligned}$$

since

$$Jf(x, y)(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & -2y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ 2xu - 2yv \end{pmatrix}.$$

8. THE ALGEBRA OF DIFFERENTIAL FORMS

Let M be a smooth manifold of dimension n and let $p \in M$. The *cotangent space* to M at p is the dual space $T_p^*M = \{f: T_pM \rightarrow \mathbb{R} \mid f \text{ linear}\}$. We define the *cotangent bundle* for M using the same procedure we used for the tangent bundle. Start by defining T^*M as the disjoint union:

$$T^*M := \bigsqcup_{p \in M} T_p^*M.$$

To define the manifold/vector bundle structure on T^*M , for each chart (U, h) on M we chose the standard basis $(\partial/\partial x_i)_p$ for $i = 1, \dots, n$ for tangent space with respect to that chart at each point $p \in U$. The gluing instructions were provided by the linear mappings $D_p(k \circ h^{-1}): \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $k \circ h^{-1}$ is a transition function for M . For the cotangent bundle, we choose the dual basis $dx_{i,p} := (\partial/\partial x_i)_p^*$. Thus,

$$dx_{i,p} \left(\left(\frac{\partial}{\partial x_j} \right)_p \right) = \delta(i, j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

The gluing instructions for the bundle are provided by the dual mapping $D_p(k \circ h^{-1})^*$. We can take the charts for T^*M so be the same as those for TM by identifying T_pM and T_p^*M in local coordinates via the mapping $(\partial/\partial x_i)_p \mapsto dx_{i,p}$. Then, if (U, h) and (V, k) are overlapping charts on M , the corresponding transition function for T^*M is given by

$$\begin{array}{ccc} & \pi^{-1}(U \cap V) & \\ \tilde{h} \swarrow & & \searrow \tilde{k} \\ h(U \cap V) \times \mathbb{R}^n & \xleftarrow{(h \circ k^{-1}, D(k \circ h^{-1})^*)} & k(U \cap V) \times \mathbb{R}^n \\ & (h(p), J(k \circ h^{-1})^T(p)y) \xleftarrow{\quad \quad \quad} (k(p), y). & \end{array}$$

Similarly, we can define vector bundles of the form $\text{Sym}^\ell TM$, $\Lambda^\ell TM$, $\Lambda^\ell T^*M$, $TM \otimes T^*M$, etc. For the special case of $\Lambda^n T^*M$, where $n = \dim M$, the transition function for charts (U, h) and (V, k) is as displayed above for T^*M except that $\mathbb{R}^n$ is replaced by $\mathbb{R} \approx \Lambda^n \mathbb{R}^n$ and $J(k \circ h^{-1})^T(p)$ is replaced by $\det(J(k \circ h^{-1})^T(p)) = \det(J(k \circ h^{-1})(p))$, i.e., multiplication by the determinant of the transition function at each point p .

Definition 8.1. A k -form on M is a global section of $\Lambda^k T^*M$, the vector bundle over M where the fiber over $p \in M$ is $\Lambda^k T_p^*M$, the k -th exterior power of the cotangent space at p . We use $\Omega^k M$ to denote the vector space of k -forms.

Fixing a chart (U, h) and bases as above, a basis for $\Lambda^k T_p^*M$ is provided by

$$\{dx_{\mu,p} := dx_{\mu_1,p} \wedge \cdots \wedge dx_{\mu_k,p} \mid 1 \leq \mu_1 < \cdots < \mu_k \leq n\},$$

where the subscript p is sometimes dropped for convenience. Now consider a k -form $\omega: M \rightarrow \Lambda^k T^*M$. In local coordinates we get

$$\omega(p) = \sum_{\mu} \omega_{\mu}(p) dx_{\mu} = \sum_{\mu} \omega_{\mu}(p_1, \dots, p_n) dx_{\mu_1} \wedge \dots \wedge dx_{\mu_k} \in \Lambda^k T_p^*M,$$

where each function $\omega_{\mu}: h(U) \rightarrow \mathbb{R}$ is differentiable. (In the above displayed equation, we are abusing notation slightly by using p to denote what really should be $h(p)$.)

We note that in the special case where $k = n = \dim M$, then changing basis affects the local form of ω via the determinant of the transition function.

8.1. The pullback of a differential form by a smooth map. A smooth map $f: M \rightarrow N$ of manifolds induces a map $\Omega^k f: \Omega^k N \rightarrow \Omega^k M$ of k -forms. To describe this map in coordinates, let us fix some $p \in M$ and pick charts (U, h) at p and (V, k) at $f(p)$ such that $f(U) \subseteq V$.

First note that $f: M \rightarrow N$ induces a map of tangent spaces $df_p: T_p M \rightarrow T_p N$. Taking its dual gives us a map of cotangent spaces (in the opposite direction) $df_p^*: T_p^* N \rightarrow T_p^* M$.

To be more specific, let us start with the case $N = \mathbb{R}$ and suppose that $g: M \rightarrow \mathbb{R}$ is differentiable. We can decompose the induced map $dg_p: T_p M \rightarrow T_{g(p)} \mathbb{R} \cong \mathbb{R}$ as a linear combination of the standard basis elements of $T_p^* M$ using the following push-forward mapping:

$$\begin{array}{ccc} T_p M & \xrightarrow{g_{*,p}=dg_p} & T_{g(p)} \mathbb{R} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R}^n & \xrightarrow{\nabla g(p)} & \mathbb{R} \end{array}$$

A basis vector $(\frac{\partial}{\partial x_i})_p$ of $T_p M$ is sent to e_i , the i -th standard basis vector of $\mathbb{R}^n$, which is then sent to $\frac{\partial g}{\partial x_i}(p) = \frac{\partial \text{id}_{\mathbb{R}} \circ g \circ h^{-1}}{\partial x_i}(h(p))$. Thus $dg_p \in T_p^* M$ and in coordinates, we can simply write

$$dg_p = \sum_{i=1}^n \left(\frac{\partial g}{\partial x_i} \right) (p) dx_{i,p}.$$

Example 8.2. For example, if we have $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ with $g(x, y) = xy^2 + y$, then at $p = (1, 1)$, $dg = y^2 dx + (2xy + 1) dy$ and $dg_p = dx + 3 dy$.

Now let us return to the function we started with, a smooth map $f: M \rightarrow N$ of manifolds. Suppose that M has dimension m and N has dimension n . What does the previous case say about the induced map df_p^* ? Again recall the diagram for df_p :

$$\begin{array}{ccccc} (\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_m})_p & & T_p M & \xrightarrow{df_p} & T_p N & & (\frac{\partial}{\partial y_1})_p, \dots, (\frac{\partial}{\partial y_n})_p \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ e_1, \dots, e_m & & \mathbb{R}^m & \xrightarrow[\substack{Jf_p \\ (k \circ f \circ h^{-1})'(h(p))}]{Jf_p} & \mathbb{R}^n & & \tilde{e}_1, \dots, \tilde{e}_n \end{array}$$

Taking its dual gives us the following diagram:

$$\begin{array}{ccccc}
dy_{1,f(p)}, \dots, dy_{n,f(p)} & T_{f(p)}^* N & \xrightarrow{df_p^*} & T_p^* M & dx_{1,p}, \dots, dx_{m,p} \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
\tilde{e}_1^*, \dots, \tilde{e}_n^* & (\mathbb{R}^n)^* & \longrightarrow & (\mathbb{R}^m)^* & e_1^*, \dots, e_m^* \\
\downarrow & \cong \downarrow & & \downarrow \cong & \downarrow \\
\tilde{e}_1, \dots, \tilde{e}_n & \mathbb{R}^n & \xrightarrow{Jf_p^\top} & \mathbb{R}^m & e_1, \dots, e_m
\end{array}$$

This means that $df_p^*(dy_{i,p})$ is the i -th column of the matrix $Jf_p^\top$, which is the i -th row of the Jacobian Jf_p . In this way we have

$$df_p^*(dy_{i,p}) = \nabla f_i(p) = \sum_{j=1}^m \frac{\partial f_i}{\partial x_j}(p) dx_j = d(f_i)_p \stackrel{\text{call}}{=} df_{i,p}.$$

It follows that df_p^* induces a map

$$\begin{aligned}
\Lambda^k df_p^* : \Lambda^k T_{f(p)}^* N &\longrightarrow \Lambda^k T_p^* M \\
dy_{\mu,p} &\longmapsto df_{\mu_1,p} \wedge \dots \wedge df_{\mu_k,p} \stackrel{\text{call}}{=} df_{\mu,p}
\end{aligned}$$

Now take a k -form $\omega : N \rightarrow \Lambda^k T^* N$ on N . Define

$$\begin{aligned}
\Omega^k f_p \omega &\stackrel{\text{call}}{=} f_p^* \omega := \Lambda^k df_p^* \circ \omega \circ f : M \longrightarrow \Lambda^k T^* M \\
p &\longmapsto \sum_{\mu} \omega_{\mu}(f(p)) df_{\mu,p},
\end{aligned}$$

where $\omega_{\mu}(f(p))$ is the coefficient of $dy_{\mu,p}$ in ω . Finally, we can glue everything together and obtain a corresponding k -form $f^* \omega : M \rightarrow \Lambda^k T^* M$ on M as shown in the following diagram.

$$\begin{array}{ccc}
\Lambda^k T^* M & \xleftarrow{\Lambda^k df^*} & \Lambda^k T^* N \\
f^* \omega \uparrow & & \uparrow \omega \\
M & \xrightarrow{f} & N
\end{array}$$

And locally, we have

$$f_p^* \left(\sum_{\mu} w_{\mu}(f(p)) dy_{\mu,f(p)} \right) = \sum_{\mu} w_{\mu}(f(p)) df_{\mu,p}.$$

Example 8.3. Define

$$\begin{aligned}
f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\
(u, v) &\longmapsto (u^2 - v, u + 2v, v^2)
\end{aligned}$$

and consider $\omega = x^2 dx \wedge dy + (x + z) dy \wedge dz \in \Omega^2 \mathbb{R}^3$. We can compute $f^* \omega$:

$$\begin{aligned}
(f^* \omega)(du \wedge dv) &= \omega_{x,y}(f(u, v)) df_{x,y} + \omega_{y,z}(f(u, v)) df_{y,z} \\
&= (u^2 - v)^2 d(u^2 - v) \wedge d(u + 2v) + (u^2 - v + v^2) d(u + 2v) \wedge d(v^2) \\
&= (u^4 - 2u^2 v + v^2)(2u du - dv) \wedge (du + 2dv) \\
&\quad + (u^2 - v + v^2)(du + 2dv) \wedge (2v dv) \\
&= \dots
\end{aligned}$$

$$= (4u^5v + u^4 - 8u^3v^2 + 4uv^3 + 2v^3 - v^2)du \wedge dv.$$

Exercise 8.4. Note that there is an isomorphism $\Lambda^1 T^*M \cong T^*M$. So for ω a 1-form on M and any point $p \in M$, $\omega(p)$ is the same as a smooth, real-valued function on the tangent space T_pM . Consider a differentiable map $f: \mathbb{R}^n \rightarrow \mathbb{R}$. Let $\omega = dx \in \Omega^1\mathbb{R}$. What is the pullback $f^*\omega$?

8.2. The exterior derivative. The next goal is to associate to each manifold M a cochain complex

$$0 \rightarrow \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \Omega^3 M \xrightarrow{d} \dots$$

called the *de Rham complex*. For each $k \geq 0$, we will construct a linear operator $d: \Omega^k M \rightarrow \Omega^{k+1} M$, called the *exterior derivative*, such that $d^2 = 0$ (and d behaves well under change of coordinates). The exterior derivative and the product on exterior products discussed earlier (via concatenation) makes $\Omega^\bullet M := \bigoplus_{k \geq 0} \Omega^k M$ into what is called a graded, anti-commutative, differential algebra.

Theorem 8.5. *Let M be a manifold of dimension n . There exists a unique sequence of linear maps*

$$(3) \quad 0 \rightarrow \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \Omega^3 M \xrightarrow{d} \dots$$

such that:

- (i) *If $f \in \Omega^0 M$, i.e., $f: M \rightarrow \mathbb{R}$, then df is the normal differential;*
- (ii) *Sequence (3) is a complex, i.e., $d^2 = d \circ d = 0$;*
- (iii) *d satisfies the product rule: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$ for $\omega \in \Omega^r M$.*

In local coordinates, d is given by

$$(4) \quad d\left(\sum_{\mu} a_{\mu} dx_{\mu}\right) = \sum_{\mu} \sum_{i=1}^n \frac{\partial a_{\mu}}{\partial x_i} dx_i \wedge dx_{\mu}.$$

Proof. Suppose that $d: \Omega^k M \rightarrow \Omega^{k+1} M$ is a linear map satisfying the above three conditions. We first prove uniqueness of d by showing that it has to be given by equation (4) in any local coordinates.

Let ω be a k -form and suppose that $\omega = \sum_{\mu} \omega_{\mu} dx_{\mu}$ using local coordinates on U . Using all three properties of d , we have

$$\begin{aligned} d\omega &= d\left(\sum_{\mu} \omega_{\mu} dx_{\mu}\right) \\ &= \sum_{\mu} (d(\omega_{\mu} dx_{\mu})) \\ &= \sum_{\mu} (d(\omega_{\mu}) \wedge dx_{\mu} + (-1)^0 \omega_{\mu} (d(dx_{\mu}))) \\ &= \sum_{\mu} \left(\left(\sum_{i=1}^n \frac{\partial \omega_{\mu}}{\partial x_i} dx_i \wedge dx_{\mu} \right) + 0 \right) \\ &= \sum_{\mu} \sum_{i=1}^n \frac{\partial \omega_{\mu}}{\partial x_i} dx_i \wedge dx_{\mu}. \end{aligned}$$

Now we show that the definition in equation (4) satisfies the three properties, proving existence of such an operator. Note that df being the normal differential follows immediately from the definition. To see that $df^2 = 0$, without loss of generality, consider the form $a dx_\mu$ and compute:

$$d^2(a dx_\mu) = d\left(\sum_{j=1}^n \frac{\partial a}{\partial x_j} dx_j \wedge dx_\mu\right) = \sum_{i,j=1}^n \frac{\partial^2 a}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_\mu.$$

If $i = j$, then $dx_i \wedge dx_j = 0$. If not, then $\frac{\partial^2 a}{\partial x_i \partial x_j} dx_i \wedge dx_j = -\frac{\partial^2 a}{\partial x_j \partial x_i} dx_j \wedge dx_i$.

To show that df satisfies the product rule, first note that for any zero-form $f \in \Omega^0 M$, by definition we have

$$d(f dx_\mu) = (df) \wedge dx_\mu.$$

Now consider a k -form $\omega = f dx_\mu$ and any other differential form $\eta = g dx_\nu$. Using the usual product rule, we compute:

$$\begin{aligned} d(\omega \wedge \eta) &= d(f dx_\mu \wedge g dx_\nu) \\ &= d(fg dx_\mu \wedge dx_\nu) \\ &= d(fg) \wedge dx_\mu \wedge dx_\nu \\ &= (g df + f dg) \wedge dx_\mu \wedge dx_\nu \\ &= (df \wedge dx_\mu) \wedge (g dx_\nu) + (-1)^k (f dx_\mu) \wedge (dg \wedge dx_\nu), \end{aligned}$$

where the $(-1)^k$ comes from $dg \wedge dx_\mu = (-1)^k dx_\mu \wedge dg$.

We have now shown that for each chart (U, h) in an atlas for M , we have an operator d_U satisfying the required properties. On overlaps, these operators must agree by uniqueness, and hence they glue together to define a global operator d . $\square$

Definition 8.6. The operator df is called *exterior differentiation*, and $d\omega$ is called the *exterior derivative* of ω .

Lemma 8.7. Let $f: M \rightarrow N$ be a smooth map of manifolds. Then f^* , the pullback by f , commutes with d . That is, for any $\omega \in \Omega^k N$, we have

$$f^*(d\omega) = d(f^*\omega).$$

Exercise 8.8. Prove the above lemma.

The fact that

$$0 \rightarrow \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \Omega^3 M \xrightarrow{d} \dots$$

is a sequence of mappings of abelian groups (even of vector spaces) and that $d^2 = 0$ says that the sequence forms a *cochain complex*.⁵ A mapping of cochain complexes is a sequence of homomorphisms between spaces with the same indices and commuting with the corresponding d mappings. For instance, Lemma 8.7 tells us that the pullback is a cochain map. In detail, given $f: M \rightarrow N$ we get the following commutative diagram:

⁵A *chain* would be a sequence of such mappings between abelian groups in which the indices on the spaces went down instead of up.

$$\begin{array}{ccccccc}
0 & \xrightarrow{d} & \Omega^0 N & \xrightarrow{d} & \Omega^1 N & \xrightarrow{d} & \Omega^2 N \xrightarrow{d} \dots \\
f^* \downarrow & & f^* \downarrow & & f^* \downarrow & & f^* \downarrow \\
0 & \xrightarrow{d} & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \Omega^2 M \xrightarrow{d} \dots
\end{array}$$

We might abbreviate the above by $f^*: \Omega^\bullet N \rightarrow \Omega^\bullet M$.

Lemma 8.9. *Pullbacks respect compositions: applying pullbacks to a commutative diagram of mappings of manifolds*

$$\begin{array}{ccc}
M & \xrightarrow{f} & N \\
& \searrow g \circ f & \downarrow g \\
& & P
\end{array}$$

induces, for each $k \geq 0$, the commutative diagram

$$\begin{array}{ccc}
\Omega^k M & \xleftarrow{f^*} & \Omega^k N \\
& \nwarrow f^* \circ g^* & \uparrow g^* \\
& & \Omega^k P
\end{array}$$

Together with the fact that the pullback by the identity function on a manifold is the identity on k -forms, we have just seen that pullback of differential forms gives a contravariant functor from the category of smooth manifolds to the category of cochain complexes.

Definition 8.10. A k -form ω is *exact* if there is a $(k-1)$ -form η such that $d\eta = \omega$. It is *closed* if $d\omega = 0$.

Note that an exact form is closed since $d^2 = 0$. We will have much more to say about exactness when we discuss DeRham cohomology, later.

9. ORIENTED MANIFOLDS

In ordinary integration of a one-variable function, we have

$$\int_0^1 f dx = - \int_1^0 f dx.$$

This formula is a first hint at the role played by an orientation of the underlying manifold. In ordinary integration in several variables, reordering the standard basis vectors gives a change of basis whose Jacobian is the corresponding permutation matrix. Thus, via the change of variables formula for integration, the sign of the integral will change by the sign of this permutation. Thus, before we can talk about (coordinate-free) integration on a manifold, we need to add the additional structure of an *orientation*.

Definition 9.1. Let V be a real vector space. Two ordered bases $\langle v_1, \dots, v_n \rangle$ and $\langle w_1, \dots, w_n \rangle$ have the *same orientation* if the mapping $V \rightarrow V$ that sends v_i to w_i has positive determinant.

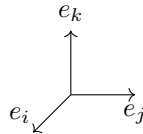
The property of having the same orientation defines an equivalence relation on the set of ordered bases of V with two equivalence classes, each of which is called an *orientation* on V . Having chosen an orientation $\mathcal{O}$, we get an *oriented vector space* $(V, \mathcal{O})$.

Ordered bases in $\mathcal{O}$ are said to be *positively oriented*; otherwise they are *negatively oriented*.

Example 9.2. Let $V = \mathbb{R}^3$ and let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$. The six possible ordered bases formed from these vectors fall into two equivalence classes:

$$\begin{aligned} \langle e_1, e_2, e_3 \rangle &\sim \langle e_2, e_3, e_1 \rangle \sim \langle e_3, e_1, e_2 \rangle, \\ \langle e_2, e_1, e_3 \rangle &\sim \langle e_1, e_3, e_2 \rangle \sim \langle e_3, e_2, e_1 \rangle. \end{aligned}$$

Exercise 9.3. Draw the six frames of the form



for the vectors given in the previous example. How is orientation reflected in the geometry of these frames.

Note that if we have $\mathbb{R}^n$ with the standard basis $e_1, \dots, e_n$, then an ordered basis formed using these vectors can be identified with a permutation $\sigma \in S_n$ on n letters. In this way, the ordered basis $\langle e_{\sigma(1)}, \dots, e_{\sigma(n)} \rangle$ has the same orientation as $\langle e_1, \dots, e_n \rangle$ if $\text{sign}(\sigma) = 1$.

Also, for example, $\langle 2e_1 + e_2, e_2, e_3 \rangle \sim \langle e_1, e_2, e_3 \rangle$ since

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 > 0.$$

In order to integrate on manifolds, we want the tangent spaces to be oriented in a coherent way.

Definition 9.4. Let M be a manifold of dimension n . A family $\{\mathcal{O}_p\}_{p \in M}$ of orientations $\mathcal{O}_p$ of the tangent space $T_p M$ is *locally coherent* if around every point of M there is an *orientation-preserving* chart (U, h) , meaning that for every $u \in U$, the differential

$$dh_u: T_u M \xrightarrow{\cong} \mathbb{R}^n$$

takes the orientation $\mathcal{O}_u$ to the usual orientation of $\mathbb{R}^n$ (the orientation of $\langle e_1, \dots, e_n \rangle$).

Definition 9.5. An *orientation* of a manifold M is a locally coherent family $\mathcal{O} = \{\mathcal{O}_p\}_{p \in M}$ of orientations of its tangent spaces. An *oriented manifold* is a pair $(M, \mathcal{O})$ consisting of a manifold M and an orientation $\mathcal{O}$ of M .

Definition 9.6. A diffeomorphism $f: M \rightarrow N$ is *orientation-preserving* if $df_p: T_p M \rightarrow T_{f(p)} N$ is orientation-preserving for all $p \in M$.

Definition 9.7. An atlas $\mathfrak{A}$ of M is called an *orienting atlas* if all its transition maps w are orientation-preserving, that is, if $\det(Jw(x)) > 0$ for all appropriate x .

Remark 9.8. If M is already oriented, then the orientation of M gives a maximal orienting atlas. Conversely, if $\mathfrak{A}$ is an orienting atlas, then there is a unique orientation of M such that $\mathfrak{A}$ consists of orientation-preserving charts. (Note that “orientation-preserving” means different things for charts and for diffeomorphisms.)

Exercise 9.9. Convince yourself that the above remark is true.

Example 9.10. The n -sphere is orientable. To see this, we use charts given by *stereographic projection*. For example, we look at the circle S^1 centered at the origin $(0, 0)$ as shown in Figure 14, with the arrow indicating the chosen orientation (counterclockwise). Let x, y denote the coordinates of $\mathbb{R}^2$. We can either project S^1

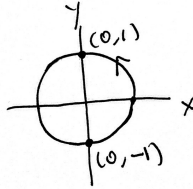


FIGURE 14. S^1 with the counterclockwise orientation.

onto the line $y = -1$ from the north pole $(0, 1)$ or project S^1 onto the line $y = 1$ from the south pole $(0, -1)$ as shown in Figure 15. In this way we obtain two charts covering S^1 . Let $U^+ = S^1 \setminus \{(0, 1)\}$, $U^- = S^1 \setminus \{(0, -1)\}$, and let ϕ^+ and ϕ^- denote the corresponding projection maps. Note that both $\phi^+(U^+)$ and $\phi^-(U^-)$ are isomorphic to $\mathbb{R}$ and the transition map from U^+ to U^- is smooth (check this!). However, the transition map is *not* orientation-preserving. Notice that the orientation of the image $\phi^-(U^-)$ is different from that of $\phi^+(U^+)$. Hence the charts

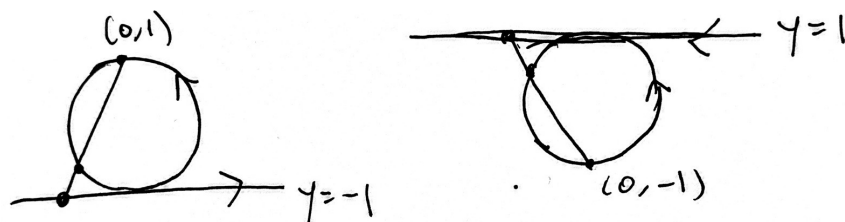


FIGURE 15. Stereographic projection of S^1 from the north pole and from the south pole.

(U^+, ϕ^+) , (U^-, ϕ^-) do not form an orienting atlas. To resolve the problem, we can change ϕ^- to $-\phi^-$ and use the charts (U^+, ϕ^+) , $(U^-, -\phi^-)$ instead.

Example 9.11. The open Möbius strip in Example 6.5 is non-orientable. The projective space $\mathbb{P}^n$ is orientable if and only if n is odd, e.g. $\mathbb{P}^1 \approx S^1$. We will later develop better tools for proving non-orientability.

10. INTEGRATION OF FORMS

For a quick introduction to the Lebesgue integral, see Appendix C.

Let M be an oriented manifold of dimension n and let $\omega \in \Omega^n M$ be an n -form. In order to integrate ω on M and compute the integral $\int_M \omega$, we want to break M into nice “cells” and integrate ω on these cells with respect to local coordinates.

10.1. Definition and properties of the integral.

Definition 10.1. A subset $X \subseteq M$ is *measurable* if $h(X \cap U) \subseteq \mathbb{R}^n$ is measurable for all charts (U, h) . We say that X has *measure zero* if $h(U \cap X) \subseteq \mathbb{R}^n$ has measure zero for all charts (U, h) .

To define the integral, we cover M by a countable collection of disjoint measurable sets $\{A_i\}_i$ and orientation-preserving charts $\{(U_i, h_i)\}_i$ with $A_i \subseteq U_i$. (Recall that M is second countable.) Locally, with respect to each (U_i, h_i) , we have

$$\omega(p) = \tilde{a}_i(p) dx_{1,p} \wedge \cdots \wedge dx_{n,p}$$

for some $\tilde{a}_i: U_i \rightarrow \mathbb{R}$. Define $a_i: h_i(U_i) \rightarrow \mathbb{R}$ by $a_i = \tilde{a}_i \circ h_i^{-1}$.

Remark 10.2. To see that the sets $\{A_i\}$ and $\{U_i\}$ with the claimed properties always exist, start with any orienting atlas $\mathfrak{V} = \{(V_\alpha, k_\alpha)\}$ (this is possible since we are assuming M is orientable). Since M is second-countable, it has a countable basis $\mathcal{B}$ for its topology. For each $p \in M$, take a chart (V_α, k_α) at p , and then, since $\mathcal{B}$ is a basis, we can find $U \in \mathcal{B}$ such that $p \in U \subseteq V_\alpha$. Save the chart (U, h) where $h := k_\alpha|_U$. In the end, since $\mathcal{B}$ is countable, so is the set of charts we saved. Thus, we have new countable orienting atlas $\mathfrak{U} = \{(U_i, h_i)\}$. Now let $A_1 := U_1$, and let $A_{i+1} := U_{i+1} \setminus \cup_{j=1}^i A_j$ for $i \geq 1$.

Definition 10.3. Using the notation above, we say that ω is *integrable* if each $a_i: h_i(U_i) \rightarrow \mathbb{R}$ is integrable on $h_i(A_i)$ and if $\sum_i \int_{h_i(A_i)} |a_i| < \infty$. In this case, we define the integral to be the sum:

$$\int_M \omega := \sum_i \int_{h_i(A_i)} a_i.$$

Theorem 10.4. Let M be a manifold of dimension n and let $\omega \in \Omega^n M$ be integrable. Then $\int_M \omega$ is independent of choice of charts.

Proof. Suppose that there is another countable collection $\{B_j\}_j$ of disjoint measurable sets that cover M and that there are orientation-preserving charts $\{(V_j, k_j)\}$ such that $B_j \subseteq V_j$. Locally, with respect to each (V_j, k_j) , write

$$\omega = \tilde{b}_j dy_1 \wedge \cdots \wedge dy_n$$

for some $\tilde{b}_j: V_j \rightarrow \mathbb{R}$ and define $b_j: k_j(V_j) \rightarrow \mathbb{R}$ by $b_j = \tilde{b}_j \circ k_j^{-1}$. We want to show that: (1) b_j is integrable on $k_j(B_j)$; (2) $\sum_j \int_{k_j(B_j)} |b_j| < \infty$; and (3) $\sum_i \int_{h_i(A_i)} a_i = \sum_j \int_{k_j(B_j)} b_j$.

First we look at the intersection $A_i \cap B_j$ and explicitly apply the change-of-variable theorem. Let $\phi_{i,j} = h_i \circ k_j^{-1}$ be the transition function, and recall that we have the following diagrams:

$$\begin{array}{ccccc}
& T_p M & & T_p^* M & \\
(dk_j)_p \swarrow & & (dh_i)_p \searrow & \xrightarrow{\text{dualizing}} & (dh_i)_p^* \swarrow & & (dk_j)_p^* \searrow \\
\mathbb{R}^n & \xrightarrow{(J\phi_{i,j})_{k_j(p)}} & \mathbb{R}^n & & \mathbb{R}^n & \xleftarrow{(J\phi_{i,j})_{k_j(p)}^\top} & \mathbb{R}^n
\end{array}$$

Now take the n -th exterior power, and the transition map becomes multiplication by $\det(J\phi_{i,j}(k_j(p)))$. Since $\Lambda^n \mathbb{R}^n \cong \mathbb{R}$, we have the following diagram:

$$\begin{array}{ccccc}
& dy_{1,p} \wedge \cdots dy_{n,p} & \Lambda^n T_p^* M & dx_{1,p} \wedge \cdots dx_{n,p} & \\
1 \swarrow & & \nwarrow & & \swarrow \\
& \mathbb{R} & \xleftarrow{\cdot \det((J\phi_{i,j})_{k_j(p)}^\top) = \cdot \det((J\phi_{i,j})_{k_j(p)})} & \mathbb{R} & \\
& & & & 1
\end{array}$$

This means that

$$dx_{1,p} \wedge \cdots \wedge dx_{n,p} = \det((J\phi_{i,j})_{k_j(p)}) dy_{1,p} \wedge \cdots \wedge dy_{n,p}.$$

Also, that for any $p \in A_i \cap B_j$, we have

$$\omega(p) = \tilde{a}_i(p) dx_{1,p} \wedge \cdots \wedge dx_{n,p} = \tilde{b}_j(p) dy_{1,p} \wedge \cdots \wedge dy_{n,p},$$

which gives us

$$\tilde{a}_i(p) \det((J\phi_{i,j})_{k_j(p)}) = \tilde{b}_j(p).$$

Therefore,

$$\begin{aligned}
b_j(k_j(p)) &= \tilde{b}_j(p) \\
&= \tilde{a}_i(p) \det((J\phi_{i,j})_{k_j(p)}) \\
&= a_i(h_i(p)) \det((J\phi_{i,j})_{k_j(p)}) \\
&= (a_i \circ \phi_{i,j})(k_j(p)) \det((J\phi_{i,j})_{k_j(p)}),
\end{aligned}$$

i.e.,

$$b_j = (a_i \circ \phi_{i,j}) \cdot \det(J\phi_{i,j}).$$

Now compute, applying the change of variables theorem:

$$\begin{aligned}
\int_M \omega &= \sum_i \int_{h_i(A_i)} a_i = \sum_i \sum_j \int_{h_i(A_i \cap B_j)} a_i \\
&= \sum_i \sum_j \int_{(\phi_{i,j} \circ k_j)(A_i \cap B_j)} a_i \\
&= \sum_i \sum_j \int_{k_j(A_i \cap B_j)} (a_i \circ \phi_{i,j}) \cdot \det(J\phi_{i,j}) \\
&= \sum_j \sum_i \int_{k_j(A_i \cap B_j)} b_j \\
&= \sum_j \int_{k_j(B_j)} b_j.
\end{aligned}$$

Note that $\det(J\phi_{i,j}) > 0$ since we are using an orienting atlas. $\square$

Definition 10.5. The *support* of an n -form ω is the closed set

$$\text{supp}(\omega) := \overline{\{p \in M \mid \omega_p \neq 0\}} \subseteq M.$$

Remark 10.6. If M itself is compact, then *all* n -forms have compact support.

Exercise 10.7. Convince yourself that an n -form $\omega \in \Omega^n M$ with compact support is integrable if and only if it is locally integrable, meaning that around any point, there is a chart (U, h) such that $\omega \circ h^{-1}: h(U) \rightarrow \mathbb{R}$ is integrable on $h(U) \subseteq \mathbb{R}^n$. Therefore, if M is compact, then all n -forms are integrable.

Exercise 10.8. Let $-M$ be M with the opposite orientation and let $\omega \in \Omega^n M$ be integrable. Show that $\int_{-M} \omega = \int_M -\omega$.

Proposition 10.9. Let M and N be manifolds of dimension n . Consider $f: M \rightarrow N$ an orientation-preserving diffeomorphism. If $\omega \in \Omega^n N$ is integrable, then the pullback $f^*\omega$ is integrable on M and

$$\int_M f^*\omega = \int_N \omega.$$

Proof. Exercise. □

10.2. Manifolds with boundary. Now the goal is to prove Stokes' theorem on oriented manifolds:

$$\int_{\partial M} \omega = \int_M d\omega$$

for some $(n-1)$ -form ω with compact support. But to do so, we need a good definition of ∂M , the *boundary* of M and look at *manifolds with boundary*.

Definition 10.10. Let $\mathbb{R}_-^n$ denote the half space

$$\mathbb{R}_-^n := \{x \in \mathbb{R}^n \mid x_1 \leq 0\}$$

with the subspace topology inherited from $\mathbb{R}^n$. Define its *boundary* to be

$$\partial \mathbb{R}_-^n := \{x \in \mathbb{R}^n \mid x_1 = 0\}.$$

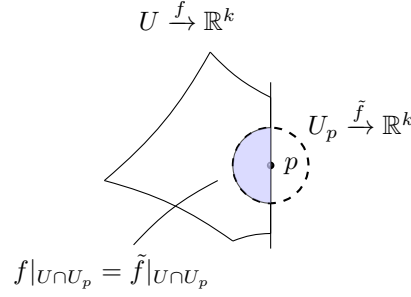
For $U \subseteq \mathbb{R}_-^n$, define the *boundary* of U to be its intersection with $\partial \mathbb{R}_-^n$

$$\partial U := U \cap \partial \mathbb{R}_-^n = U \cap (\{0\} \times \mathbb{R}^{n-1}).$$

Remark 10.11. First note that it is possible for $\partial U = \emptyset$. Also, the boundary ∂U is different from the boundary of U in the topological sense, i.e., from $U \setminus U^\circ$ (cf. Definition B.1).

Definition 10.12. Let $U \subseteq \mathbb{R}_-^n$ be open. A map $f: U \rightarrow \mathbb{R}^k$ is *differentiable* at $p \in U$ if there exist an open neighborhood $U_p \subseteq \mathbb{R}_-^n$ of p and $\tilde{f}: U_p \rightarrow \mathbb{R}^k$ a differentiable map (in the normal sense) such that $f|_{U \cap U_p} = \tilde{f}|_{U \cap U_p}$. We call $\tilde{f}$ a *local extension* of f .

Definition 10.13. Let $U, V \subseteq \mathbb{R}_-^n$ be open. A differentiable map $f: U \rightarrow V$ is a *diffeomorphism* if it is bijective and has a differentiable inverse.

FIGURE 16. Differentiability at ∂U .

Lemma 10.14. *Let $U, V \subseteq \mathbb{R}^n$ be open and let $f: U \rightarrow V$ be a diffeomorphism. Then $f(\partial U) = \partial V$. Hence, $f|_{\partial U}: \partial U \xrightarrow{\cong} \partial V$ is a diffeomorphism between open sets of $\mathbb{R}^{n-1}$.*

Proof. Let $p \in \partial U$, and let $\tilde{f}: U_p \rightarrow \mathbb{R}^n$ be a local extension of f . Suppose for contradiction that $f(p) \in V \setminus \partial V$.

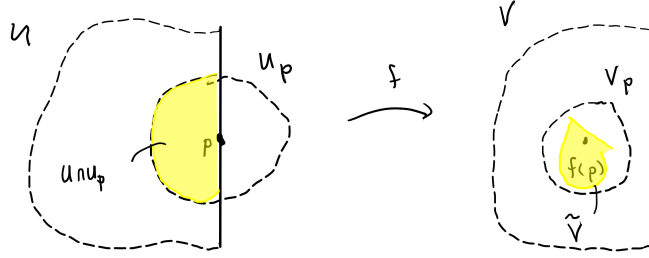


FIGURE 17. What's wrong with this picture?

Since f^{-1} is continuous, $(f^{-1})^{-1}(U \cap U_p) = f(U \cap U_p) \subseteq V$ is open in $\mathbb{R}^n$. Let $V_p \subseteq f(U \cap U_p)$ be a neighborhood of $f(p)$ with the further restriction that $V_p \subseteq V \setminus \partial V$. Such V_p exists since $f(p) \notin \partial V$ by assumption. Define $\tilde{V} = f(U \cap U_p)$, and consider the restriction $f^{-1}|_{\tilde{V}}: \tilde{V} \rightarrow U \cap U_p \subset \mathbb{R}^n$. Our goal is to show that $U \cap U_p$, which is open in $\mathbb{R}^n$, is actually open in $\mathbb{R}^n$. That will yield a contradiction, since $U \cap U_p$ contains no open ball about p .

Since $f \circ f^{-1}|_{\tilde{V}} = \text{id}_{\tilde{V}}$, for any $x \in \tilde{V}$, we have

$$J(f \circ f^{-1})_x = Jf_{f^{-1}(x)} (Jf^{-1})_x = I_n,$$

where I_n is the identity matrix. In particular, Jf_x^{-1} is invertible for all x . By the inverse function theorem, $f^{-1}|_{\tilde{V}}$ is a local diffeomorphism (cf. Definition A.6) and, hence, an open map. Therefore, $f^{-1}(\tilde{V}) = U \cap U_p \subseteq U$ is a neighborhood of p that is open in $\mathbb{R}^n$. This contradicts the assumption that $p \in \partial U$. Thus, $f(\partial U) \subseteq \partial V$. The other inclusion follows from applying the same argument to f^{-1} . $\square$

Lemma 10.15. *Let $U, V \subseteq \mathbb{R}^n$ be open and let $f: U \rightarrow V$ be a diffeomorphism. Let $p \in \partial U$. Then the well-defined differential $df_p: \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ maps $\partial\mathbb{R}^n_-$ to $\partial\mathbb{R}^n_-$, $\mathbb{R}^n_-$ to $\mathbb{R}^n_-$, and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ to $\mathbb{R}^n_+$.*

Proof. Since we can extend f to some neighborhood of p in $\mathbb{R}^n$, the differential df_p is well-defined. The goal is to show that the Jacobian Jf_p is of the form

$$Jf_p = \left[\begin{array}{c|ccc} \frac{\partial f_1}{\partial x_1}(p) & 0 & \cdots & 0 \\ \frac{\partial f_2}{\partial x_1}(p) & & & \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1}(p) & & & \end{array} \middle| \begin{array}{c} (Jf|_{\partial\mathbb{R}^n_-})_p \end{array} \right]$$

with $\frac{\partial f_1}{\partial x_1}(p) > 0$. Then the result follows. First recall that

$$\frac{\partial f_i}{\partial x_j}(p) = \lim_{t \rightarrow 0} \frac{f_i(p + te_j) - f_i(p)}{t}.$$

By Lemma 10.14, for $j \geq 2$, we have

$$\frac{\partial f_1}{\partial x_j}(p) = \lim_{t \rightarrow 0} \frac{f_1(p + te_j) - f_1(p)}{t} = \lim_{t \rightarrow 0} \frac{0 - 0}{t} = 0.$$

So the first row of Jf_p except for the first entry is zero. At the same time, since $f_1(p + te_1) \leq 0$,

$$\frac{\partial f_1}{\partial x_1}(p) = \lim_{t \rightarrow 0^-} \frac{f_1(p + te_1) - f_1(p)}{t} = \lim_{t \rightarrow 0^-} \frac{f_1(p + te_1) - 0}{t} \geq 0.$$

This forces $\frac{\partial f_1}{\partial x_1}(p) > 0$ or otherwise $\det(Jf_p) = 0$. $\square$

Definition 10.16. An n -dimensional *manifold with boundary* is a second-countable, Hausdorff topological space M that is locally homeomorphic to open subsets of $\mathbb{R}^n$ with differentiable transition functions. A point $p \in M$ is in the *boundary* of M if there is some (hence every) chart (U, h) at p such that $h(p) \in \partial h(U) \subseteq \mathbb{R}^n_-$. The collection of all such points is denoted ∂M .

Remark 10.17. Again note the difference between ∂M and the topological boundary of M .

Remark 10.18. If M is an n -dimensional manifold with boundary, then the restrictions

$$h_{U \cap \partial M}: U \cap \partial M \rightarrow \partial h(U) \subseteq \partial\mathbb{R}^n_-$$

of charts (U, h) on M turn ∂M into an $(n - 1)$ -dimensional manifold (without boundary).

Exercise 10.19. Let M and N be n -dimensional manifolds with boundary and let $f: M \rightarrow N$ be a diffeomorphism between manifolds with boundary. Show that $f(\partial M) = \partial N$ and that the restriction $f|_{\partial M}: \partial M \rightarrow \partial N$ is a diffeomorphism.

Example 10.20. Let M be an n -dimensional manifold. Then it is an n -dimensional manifold with boundary and $\partial M = \emptyset$. To see this, note that for a chart (U, h) , we can create a finite collection of smaller charts $\{(U_i, h_i)\}_{i=1}^m$ such that $U = \bigcup_{i=1}^m U_i$

and $h_i(U_i) \subseteq \mathbb{R}_-^n \setminus \partial \mathbb{R}_-^n$. Conversely, if M is a manifold with boundary, then it is a manifold if $\partial M = \emptyset$.

Example 10.21. The closed ball $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary S^{n-1} . The cylinder $[0, 1] \times S^1$ is a manifold with boundary equal to the disjoint union of two circles.

Definition 10.22. Let M be a manifold with boundary and let $p \in \partial M$. Define the tangent space to M at p to be

$$T_p M := T_p^{\text{alg}} M \xrightarrow{\cong} T_p^{\text{phy}} M,$$

as described in section 4. Let (U, h) be a chart at p and define

$$T_p^+ M := (dh_p)^{-1}(\mathbb{R}_+^n), \quad T_p^- M := (dh_p)^{-1}(\mathbb{R}_-^n).$$

Note that this definition does not depend on the choice of charts by Lemma 10.15.

Remark 10.23. The inclusion $\partial M \hookrightarrow M$ gives an inclusion of tangent spaces at $p \in \partial M$, and in fact we have

$$T_p \partial M = T_p^+ M \cap T_p^- M.$$

Definition 10.24. Let M be a manifold with boundary and let $p \in \partial M$. We call elements in $T_p^- M \setminus T_p \partial M$ the *inward-pointing* tangent vectors, and elements in $T_p^+ M \setminus T_p \partial M$ the *outward-pointing* tangent vectors. Note that this definition can be given without embedding M into some $\mathbb{R}^N$.

The definition of an oriented manifold with boundary is the same as for ordinary manifolds. The boundary then is then orientable, but we would like to fix a convention for its orientation.

Definition 10.25. Let M be an n -dimensional oriented manifold with boundary and let $p \in \partial M$. We define the *natural orientation* on ∂M as follows: an ordered basis $\langle w_1, \dots, w_{n-1} \rangle$ for $T_p \partial M$ is positively oriented if for any outward-pointing tangent vector $v \in T_p M$, the ordered basis $\langle v, w_1, \dots, w_{n-1} \rangle$ for $T_p M$ is positively oriented in $T_p M$.

Example 10.26. Let D^3 denote the solid unit ball in $\mathbb{R}^3$ with its orientation induced by $\mathbb{R}^3$. Its boundary is $\partial D^3 = S^2$, and the natural orientation on S^2 is given by $\langle w_1, w_2 \rangle$ as shown in Figure 18.

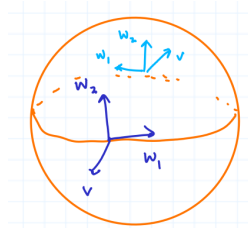


FIGURE 18. The natural orientation on $S^2 = \partial D^3$.

10.3. Stokes' theorem on manifolds.

Theorem 10.27. *Let M be an n -dimensional oriented manifold with boundary and let $\omega \in \Omega^{n-1}M$ be an $(n-1)$ -form with compact support. Let $\iota: \partial M \hookrightarrow M$ denote the inclusion and define $\int_{\partial M} \omega := \int_{\partial M} \iota^* \omega$. Then $d\omega$ is integrable and*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. We carry out the proof in three steps of increasing generality.

Case 1: $M = \mathbb{R}_-^n$. Write ω as

$$\omega = \sum_{i=1}^n a_i dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n.$$

Then

$$\begin{aligned} d\omega &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n. \end{aligned}$$

This gives us

$$\int_M d\omega = \sum_{i=1}^n \int_{\mathbb{R}_-^n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i}.$$

Use Fubini's theorem to first integrate $\int_{\mathbb{R}_-^n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i}$ with respect to x_i . For $i \neq 1$, by definition we have

$$\begin{aligned} \int_{x_i=-\infty}^{\infty} \frac{\partial a_i}{\partial x_i} &= \lim_{t \rightarrow \infty} \int_0^t \frac{\partial a_i}{\partial x_i} + \lim_{t \rightarrow -\infty} \int_t^0 \frac{\partial a_i}{\partial x_i} \\ &= \lim_{t \rightarrow \infty} (a_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) - a_i(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)) \\ &\quad + \lim_{t \rightarrow -\infty} (a_i(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) - a_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)) \\ &= (0 - a_i(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)) + (a_i(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) - 0) \\ &= 0, \end{aligned}$$

where $\lim_{t \rightarrow \infty} a_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) = 0$ since ω has compact support.

For $i = 1$, since $M = \mathbb{R}_-^n$, we have instead

$$\int_{x_1=-\infty}^0 \frac{\partial a_1}{\partial x_1} = \lim_{t \rightarrow -\infty} \int_{x_1=t}^0 \frac{\partial a_1}{\partial x_1} = a_1(0, x_2, \dots, x_n).$$

Therefore,

$$\int_M d\omega = \sum_{i=1}^n \int_{\mathbb{R}_-^n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i} = \int_{\mathbb{R}_-^n} \frac{\partial a_1}{\partial x_1} = \int_{\mathbb{R}^{n-1}} a_1(0, x_2, \dots, x_n).$$

On the other hand, note that $\iota: \partial M \rightarrow M$ is given by

$$\begin{aligned} \iota: \partial M &\longrightarrow M \\ (x_2, \dots, x_n) &\longmapsto (0, x_2, \dots, x_n). \end{aligned}$$

Since $\iota_1 \equiv 0$, we have

$$\begin{aligned}
 \int_{\partial M} \omega &= \int_{\partial M} \iota^* \omega = \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} \iota^* (a_i dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n) \\
 &= \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} a_i(0, x_2, \dots, x_n) dt_1 \wedge \cdots \wedge \hat{dt}_i \wedge \cdots \wedge dt_n \\
 &= \int_{\mathbb{R}^{n-1}} a_1(0, x_2, \dots, x_n) dx_2 \wedge \cdots \wedge dx_n \\
 &= \int_M d\omega.
 \end{aligned}$$

Case 2: Now suppose that M is an arbitrary oriented manifold with boundary and that $\text{supp}(\omega) \subseteq U$ for some orienting chart (U, h) . We can use the chart to reduce to the previous case: Using the chart and the fact that $\text{supp}(\omega) \subseteq U$, we may assume $M = U \subseteq \mathbb{R}_+^n$. Then extend ω to a form $\tilde{\omega}$ on all of $\mathbb{R}_+^n$ by letting $\tilde{\omega}|_U = \omega$, and $\tilde{\omega}|_{U^c} \equiv 0$. This will not cause any problem since $\text{supp}(\omega) \subseteq U$ is compact.

Case 3: Finally, suppose that $\omega \in \Omega^{n-1}M$ is any compactly supported form. For this case, as we describe below, we can break ω into a finite sum $\omega = \omega_1 + \cdots + \omega_r$ of $(n-1)$ -forms, each of which has compact support that is contained in a chart. Then the previous case applies and we are home.

Around each $p \in \text{supp}(\omega)$, choose an orientation-preserving chart (U_p, h_p) and a smooth and compactly-supported function $\lambda_p: M \rightarrow [0, 1]$ such that $\lambda_p(p) > 0$ and $\text{supp}(\lambda_p) \subseteq U_p$ (i.e., a bump function). Let $\tilde{U}_p := \lambda_p((0, 1]) = M \setminus \lambda_p^{-1}(0)$. Then, $\tilde{U}_p$ is open, and $\text{supp}(\omega) \subset \cup_{p \in M} \tilde{U}_p$. Since $\text{supp}(\omega)$ is compact, there exist $p_1, \dots, p_r$ such that $\text{supp}(\omega) \subset \cup_{i=1}^r \tilde{U}_{p_i} =: X$. Define r differentiable functions $\tau_1, \dots, \tau_r$ by

$$\begin{aligned}
 \tau_i: X &\longrightarrow [0, 1] \\
 x &\longmapsto \frac{\lambda_{p_i}(x)}{\sum_{j=1}^r \lambda_{p_j}(x)}.
 \end{aligned}$$

Then $\sum_{i=1}^r \tau_i(x) = 1$ for all $x \in X$, and we call $\tau_1, \dots, \tau_r$ a *partition of unity*. To find the corresponding partition of ω , define $\omega_i \in \Omega^{n-1}M$ by

$$\omega_i(p) = \begin{cases} \tau_i(p)\omega(p) & \text{if } p \in X; \\ 0 & \text{if } p \in M \setminus X. \end{cases}$$

We have that $\text{supp}(\omega_i) \subseteq \text{supp}(\lambda_{p_i}) \subset U_i$ is compact. Also, ω_i is differentiable on M and $\omega = \omega_1 + \cdots + \omega_r$. Now by the previous case, we have

$$\int_M d\omega_i = \int_{\partial M} \omega_i,$$

and finally, by linearity,

$$\int_M d\omega = \int_{\partial M} \omega.$$

□

11. DE RHAM COHOMOLOGY

Recall that there is a contravariant functor from the category of smooth manifolds to the category of cochain complexes. That is, to each manifold M , we associate the *de Rham complex*

$$0 \rightarrow \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \xrightarrow{d} \dots,$$

where at degree k we have $\Omega^k M$, the k -forms on M , and $d: \Omega^k M \rightarrow \Omega^{k+1} M$ is exterior differentiation. Furthermore, a smooth map $f: M \rightarrow N$ between manifolds induces a map between cochains

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^0 N & \xrightarrow{d} & \Omega^1 N & \xrightarrow{d} & \Omega^2 N \xrightarrow{d} \dots \\ & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ 0 & \longrightarrow & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \Omega^2 M \xrightarrow{d} \dots \end{array}$$

The cohomology groups of the de Rham complex of a manifold, defined below, are important invariants of the manifold. Thus, if two manifolds have different cohomology groups, then they are not diffeomorphic. Furthermore, using Stokes' theorem, it can be shown that de Rham cohomology is dual to singular homology on the manifold, where the latter can be computed through a triangulation of the manifold and detects topological features of the manifold, for instance, the number of k -dimensional holes (see Appendix D).

11.1. Definition and first properties. Recall that a form in the kernel of d is called *closed* and a form in the image of d is called *exact*. In the context of de Rham cohomology, closed forms are called *cocycles* and exact forms are called *coboundaries*.⁶

Since $d^2 = 0$, every exact form is closed, or said another way, every coboundary is a cocycle. Thus, we can make the following definition:

Definition 11.1. The k -th cohomology group of the de Rham complex is the quotient

$$H^k M := \ker(\Omega^k M \xrightarrow{d} \Omega^{k+1} M) / \operatorname{im}(\Omega^{k-1} M \xrightarrow{d} \Omega^k M).$$

If $\omega \in \Omega^k M$ is a cocycle, we denote the *cohomology class* of ω by

$$[\omega] := \omega + d(\Omega^{k-1} M).$$

We say that cocycles ω and η are *cohomologous* if $[\omega] = [\eta]$, i.e., if $\omega - \eta = d\alpha$ for some $\alpha \in \Omega^{k-1} M$.

The cohomology groups measure the extent to which the de Rham sequence is not exact: i.e., $H^k M = 0$ if and only if $\operatorname{im} d_{k-1} = \ker d_k$. If its dimension as an $\mathbb{R}$ -vector space is large, then the sequence is far from being exact in degree k . (See Appendix subsection D.1 for the basics on exact sequences.)

⁶The terminology of cocycles and coboundaries is motivated by singular/simplicial homology theory, where these concepts have simple geometric interpretations.

Example 11.2. Let $M = \mathbb{R}$. Since M one-dimensional, its de Rham complex is

$$0 \rightarrow \Omega^0 \mathbb{R} \xrightarrow{d} \Omega^1 \mathbb{R} \rightarrow 0.$$

Note that

$$\Omega^0 \mathbb{R} = \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is smooth}\}$$

and that

$$\Omega^1 \mathbb{R} = \{f dx \mid f: \mathbb{R} \rightarrow \mathbb{R} \text{ is smooth}\}.$$

Also, for a smooth function $f: \mathbb{R} \rightarrow \mathbb{R}$, $df = 0$ means that f is a constant function. Thus, we have $\ker(\Omega^0 \mathbb{R} \xrightarrow{d} \Omega^1 \mathbb{R}) \cong \mathbb{R}$. At the same time, for any $f \in \Omega^1 \mathbb{R}$, we can define $g(x) = \int_0^x f(t)dt$. Then g is smooth and $dg = f$. Therefore, $\text{im}(\Omega^0 \mathbb{R} \xrightarrow{d} \Omega^1 \mathbb{R}) = \Omega^1 \mathbb{R}$. In this way we can compute the cohomology groups:

$$H^0 M = \ker(\Omega^0 \mathbb{R} \xrightarrow{d} \Omega^1 \mathbb{R}) / \text{im}(0 \rightarrow \Omega^0 \mathbb{R}) \cong \mathbb{R}/0 = \mathbb{R},$$

$$H^1 M = \ker(\Omega^1 \mathbb{R} \xrightarrow{d} 0) / \text{im}(\Omega^0 \mathbb{R} \xrightarrow{d} \Omega^1 \mathbb{R}) \cong \mathbb{R}/\mathbb{R} = 0.$$

The cohomology groups are not only groups under addition, they are also $\mathbb{R}$ -vector spaces. In fact, there is even more structure. Define the *cohomology ring* of M to be

$$H^\bullet M := \bigoplus_{k \geq 0} H^k M,$$

where the product is induced by the wedge product on $\Omega^\bullet M$. This product on $H^\bullet M$ is defined to be

$$\begin{aligned} \wedge: H^r M \times H^s M &\longrightarrow H^{r+s} M \\ ([\omega], [\eta]) &\longmapsto [\omega \wedge \eta]. \end{aligned}$$

Theorem 11.3. *The product on $H^\bullet M$ is well-defined, making $H^\bullet M$ into a graded, anti-commutative $\mathbb{R}$ -algebra.⁷*

Proof. Conceptualize the cohomology groups as cocycles modulo coboundaries. We first need to check that the wedge product of two cocycles is a cocycle. Suppose ω and η are cocycles, then by the product rule for exterior differentiation, $\omega \wedge \eta$ is also a cocycle:

$$d(\omega \wedge \eta) = d\omega \wedge \eta \pm \omega \wedge d\eta = 0 + 0 = 0.$$

Next, we show that the product does not depend on the choice of representatives for $[\omega]$ and $[\eta]$. So, given $\mu \in \Omega^{r-1} M$ and $\nu \in \Omega^{s-1} M$, we must show that $[(\omega + d\mu) \wedge (\eta + d\nu)] = [\omega \wedge \eta]$. We leave this as the following exercise:

Exercise 11.4. Find $\tau \in \Omega^{r+s-1} M$ such that

$$d\tau = ((\omega + d\mu) \wedge (\eta + d\nu)) - \omega \wedge \eta.$$

□

⁷An $\mathbb{R}$ -algebra is a ring that is also naturally a vector space over $\mathbb{R}$.

Remark 11.5. Recall that a smooth map $f: M \rightarrow N$ between manifolds induces a map between k -forms $f^*: \Omega^k N \rightarrow \Omega^k M$. In fact, f also induces a group homomorphism between the k -th homology groups $f^{*,k}: H^k N \rightarrow H^k M$ that also commutes with the exterior differentiation d . In this way, $H^\bullet$ is a contravariant functor from the category of smooth manifolds to the category of graded, anti-commutative algebras over $\mathbb{R}$. We use $f^*: H^\bullet N \rightarrow H^\bullet M$ to denote the collection of induced maps $f^{*,k}$. If f is constant, then $f^{*,k} \equiv 0$ for all $k > 0$.

Proposition 11.6. *Let M be a smooth manifold of dimension n . Then $H^k M = 0$ for all $k > n$.*

Proof. Note that $\Omega^k M = 0$ for $k > n$. □

Proposition 11.7. *Suppose that M has c connected components. Then $H^0 M \cong \mathbb{R}^c$. In particular, $H^0 M \cong \mathbb{R}$ if M is connected.*

Proof. If $f \in \ker(\Omega^0 M \xrightarrow{d} \Omega^1 M)$, then locally we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = 0$, which means that $\frac{\partial f}{\partial x_i} = 0$ for all i . Therefore, f is locally constant. Since M is locally path connected, its connected components agree with its path-connected components. Then f is constant on each connected component: take a path γ in one of these components and we have $(f \circ \gamma)' \equiv 0$. (In fact, if $g: X \rightarrow Y$ is a locally constant continuous mapping of topological spaces, and points are closed in Y , e.g., Y is Hausdorff, then g is constant on connected components of X .) □

Exercise 11.8. Let $f: M \rightarrow N$ be a smooth map between manifolds. Suppose that M and N are both connected. Show that $f^{*,0}: H^0 N \rightarrow H^0 M$ is the identity map on $\mathbb{R}$.

Proposition 11.9. *Let M be an orientable manifold of dimension n . Suppose that M closed, i.e., compact with $\partial M = \emptyset$. Then $H^n M \neq 0$.*

Proof. It suffices to show that there is some n -form on M that is not exact. Orient M and choose $\omega \in \Omega^n M$ such that $\int_M \omega \neq 0$. If there were $\eta \in \Omega^{n-1} M$ such that $\omega = d\eta$, then by the Stokes' theorem, $\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$ since $\partial M = \emptyset$ by assumption. □

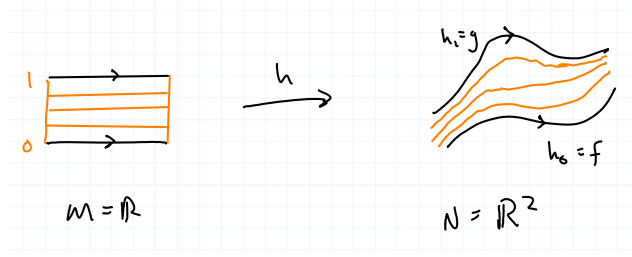
11.2. Homotopy invariance of de Rham cohomology.

Definition 11.10. Let $f, g: M \rightarrow N$ be smooth maps between manifolds. We say that f and g are (*smoothly*) *homotopic*, denoted $f \sim g$, if there is a smooth map

$$h: [0, 1] \times M \longrightarrow N$$

such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in M$ (see Figure 19). In that case, we call h a homotopy from f to g and write $f \stackrel{h}{\sim} g$.

We say that f is *null-homotopic* if f is homotopic to some constant function. We say that M is *contractible* if id_M is null homotopic.

FIGURE 19. Homotopy between mapping $f, g: \mathbb{R} \rightarrow \mathbb{R}^2$.

Example 11.11. This example shows that $\mathbb{R}^n$ is contractible. Here is a homotopy from the identity to the constant zero-mapping:

$$h: [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$(t, x) \mapsto (1 - t)x.$$

At time zero, we have $h_0(x) := h(0, x) = x = \text{id}_{\mathbb{R}^n}(x)$, and at time 1, we have $h_1(x) := h(1, x) = 0$.

Exercise 11.12. Show that $\sim$ is an equivalence relation on the set of mappings $M \rightarrow N$.

Theorem 11.13. Let $f, g: M \rightarrow N$ be smooth maps that are homotopic. Then they induce the same map on the cohomology groups, i.e.,

$$f^* = g^*: H^\bullet N \rightarrow H^\bullet M.$$

Proof. Let $h: [0, 1] \times M \rightarrow N$ denote the homotopy between f and g such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in M$. The goal is to use h to construct a collection of maps $s: \Omega^k N \rightarrow \Omega^{k-1} M$ (shown below, though the diagram does not

$$\begin{array}{ccccccc} \dots & \xrightarrow{d} & \Omega^{k-1} N & \xrightarrow{d} & \Omega^k N & \xrightarrow{d} & \Omega^{k+1} N \xrightarrow{d} \dots \\ & \searrow s & & \searrow s & & \searrow s & \\ \dots & \xrightarrow{d} & \Omega^{k-1} M & \xrightarrow{d} & \Omega^k M & \xrightarrow{d} & \Omega^{k+1} M \xrightarrow{d} \dots \end{array}$$

necessarily commute) such that for any $\omega \in \Omega^k N$,

$$(5) \quad g^* \omega - f^* \omega = (sd + ds)\omega.$$

Then if ω is closed, i.e., $d\omega = 0$, we have

$$g^* \omega - f^* \omega = (ds)\omega = d(sw),$$

which implies that $[f^* \omega] = [g^* \omega]$ and, hence, $f^* = g^*$. Thus, it suffices to show the existence the mappings s .

To reduce the problem further, for each $t \in [0, 1]$, define

$$i_t: M \longrightarrow [0, 1] \times M$$

$$x \longmapsto (t, x).$$

We have that $(h \circ i_0)(x) = h(0, x) = f(x)$ for all $x \in M$ and, similarly, $h \circ i_1 = g$. For each k , we want to define the “prism operator” $P: \Omega^k([0, 1] \times M) \rightarrow \Omega^{k-1}M$ such that

$$(6) \quad i_1^* - i_0^* = dP + Pd.$$

We can let $s := P \circ h^*$:

$$\Omega^k N \xrightarrow{h^*} \Omega^k([0, 1] \times M) \xrightarrow{P} \Omega^{k-1}M,$$

and using the fact that the exterior derivative and the pullback operators commute, we see that Equation 5 is satisfied:

$$g^* - f^* = i_1^* h^* - i_0^* h^* = (i_1^* - i_0^*) h^* = (dP + Pd) h^* = d(Ph^*) + (Ph^*)d = ds + sd,$$

as required.

So the problem becomes finding the prism operator P . Define

$$P: \Omega^k([0, 1] \times M) \longrightarrow \Omega^{k-1}M$$

$$\omega \longmapsto \int_{t=0}^1 \omega\left(\frac{\partial}{\partial t}, -\right)$$

where t is the coordinate on $[0, 1]$. We need to clarify the meaning of $\omega(\frac{\partial}{\partial t}, -)$. We are thinking of ω as a multilinear alternating form acting on tuples of tangent vectors. In local coordinates, $\partial/\partial t$ and dt are dual, i.e., $dt(\partial/\partial t) = 1$. So in local coordinates, if dt occurs in ω , by flipping signs if necessary, we make dt the leading term in the wedge product. Then $\omega(\frac{\partial}{\partial t}, -)$ is the form obtained by setting dt equal to 1. If dt does not occur, then $\omega(\frac{\partial}{\partial t}, -) = 0$.⁸ We pause here for an example of the prism operator.

Example 11.14. Let $M = \mathbb{R}^3$, and consider the $\omega \in \Omega^3([0, 1] \times \mathbb{R}^3)$ given by

$$\omega = (3t^2 + 2xt + x^2y) dt \wedge dx \wedge dy + (tx + t^2y) dx \wedge dy \wedge dz.$$

Then

$$\begin{aligned} P\omega &= P((3t^2 + 2xt + x^2y) dt \wedge dx \wedge dy) + P((tx + t^2y) dx \wedge dy \wedge dz) \\ &= P((3t^2 + 2xt + x^2y) dt \wedge dx \wedge dy) + 0 \\ &= \left(\int_{t=0}^1 (3t^2 + 2xt + x^2y) \right) dx \wedge dy \\ &= (1 + x + x^2y) dx \wedge dy \in \Omega^2M. \end{aligned}$$

A 0 appears as a summand in the second step of this calculation since dt does not appear in $(tx + t^2y) dx \wedge dy \wedge dz$.

We now resume the proof. The only thing that remains is to show that P satisfies Equation 6. This is a local question, so we check it in local coordinates $t, x_1, \dots, x_m$, where $m = \dim M$, and exterior differentiation, the pullback, and integrals are

⁸To learn more about this operation, read about *interior products* in another text on manifolds.

linear, we may assume ω has a single term. That term may or may not involve the differential dt . We consider each of those cases separately:

Case 1: Suppose that $\omega = \ell(t, x) dt \wedge dx_\mu$.

Consider the left-hand side of Equation 6 first. Pulling back by $i_0(x) = (0, x)$, we have

$$i_0^*(\omega) = \ell(0, x) d0 \wedge dx = 0.$$

Similarly, $i_1^*\omega = 0$. Therefore $i_1^*\omega - i_0^*\omega = 0$. Therefore, to verify Equation 6, we must check that $dP\omega = -Pd\omega$. Using the fact that $dt \wedge dt = 0$, compute

$$\begin{aligned} d(P\omega) &= d\left(\left(\int_{t=0}^1 \ell(t, x) dt\right) dx_\mu\right) \\ &= \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\int_0^1 \ell(x, t) dt\right) dx_i\right) \wedge dx_\mu \\ &= \left(\sum_{i=1}^n \left(\int_0^1 \frac{\partial \ell}{\partial x_i} dt\right) dx_i\right) \wedge dx_\mu \\ &= \sum_{i=1}^n \left(\int_0^1 \frac{\partial \ell}{\partial x_i} dt\right) dx_i \wedge dx_\mu, \end{aligned}$$

and

$$\begin{aligned} P(d\omega) &= P\left(\sum_{i=1}^n \frac{\partial \ell}{\partial x_i} dx_i \wedge dt \wedge dx_\mu\right) \\ &= P\left(-\sum_{i=1}^n \frac{\partial \ell}{\partial x_i} dt \wedge dx_i \wedge dx_\mu\right) \\ &= -\sum_{i=1}^n \left(\int_{t=0}^1 \frac{\partial \ell}{\partial x_i} dt\right) dx_i \wedge dx_\mu, \end{aligned}$$

as required.

Case 2: Suppose that $\omega = \ell(x, t) dx_\mu$.

Now $P\omega = 0$ and, hence, $dP\omega = 0$. We need to show that $i_1^*\omega - i_0^*\omega = Pd\omega$:

$$\begin{aligned} Pd\omega &= P\left(\frac{\partial \ell}{\partial t} dt \wedge dx_\mu + \sum_{i=1}^n \frac{\partial \ell}{\partial x_i} dx_i \wedge dx_\mu\right) \\ &= \left(\int_{t=0}^1 \frac{\partial \ell}{\partial t} dt\right) dx_\mu + 0 \\ &= (\ell(x, 1) - \ell(x, 0)) dx_\mu \\ &= i_1^*\omega - i_0^*\omega. \end{aligned}$$

□

Corollary 11.15. *If $f: M \rightarrow N$ is null-homotopic, then $f^{*,k} \equiv 0$ for all $k > 0$.*

Corollary 11.16. *If M is contractible, then $H^k M = 0$ for all $k > 0$.*

Corollary 11.17 (The Poincaré Lemma). *If $U \subseteq \mathbb{R}^n$ is open and star-shaped (meaning that there is $x \in U$ such that for all $y \in U$, the line segment connecting x and y lies entirely in U), then $H^k U = 0$ for all $k > 0$.*

Exercise 11.18. Prove the above three corollaries.

Definition 11.19. A mapping $f: M \rightarrow N$ defines a *homotopy equivalence* if there is a smooth mapping $g: N \rightarrow M$ such that $f \circ g \sim \text{id}_N$ and $g \circ f \sim \text{id}_M$. If so, we say that M and N are *homotopy equivalent*.

Remark 11.20. In particular, diffeomorphisms defines homotopy equivalences.

Theorem 11.21. *If M and N are homotopy equivalent, then $H^k M \cong H^k N$ for all k .*

Proof. This follows immediately from Theorem 11.13. $\square$

Remark 11.22 (Continuity versus smoothness.). Let M and N be smooth manifolds. We have defined homotopies for *smooth* functions $M \rightarrow N$, and further our homotopies, themselves are smooth functions. It turns out that all of our results hold if we allow our mappings to just be continuous rather than smooth. In detail: we say continuous mappings $f, g: M \rightarrow N$ (where each of M and N are still smooth manifolds) are (continuously) homotopic if there exists a continuous mapping $h: [0, 1] \times M \rightarrow N$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in M$. It can be shown that if $f: M \rightarrow N$ is a continuous mapping, then it is continuously homotopic to a smooth mapping $\tilde{f}: M \rightarrow N$, and if two smooth mapping $f, g: M \rightarrow N$ are continuously homotopic, then they are smoothly homotopic.

The reason the above results are important is that they show that de Rham cohomology is actually a topological invariant: if M and N are smooth manifolds that are homeomorphic, then there is an isomorphism of their de Rham cohomology rings. Without the above results, we could only make that conclusion if M and N were diffeomorphic—a much stronger condition.

Definition 11.23. Let $f: M \rightarrow N$ be a mapping of manifolds. Then

- (1) f is an *immersion* if $df_p: T_p M \rightarrow T_{f(p)} N$ is injective (see subsection 4.4 for the definition of df_p —locally, in terms of the “physical” version of tangent space, it is the mapping determined by the Jacobian of f);
- (2) the pair (M, f) is a *submanifold* if f is an injective immersion; if $M \subseteq N$, then we say M is a submanifold if (M, ι) is a submanifold where ι is the inclusion mapping;
- (3) f is an *embedding* if it is an injective immersion and also a homeomorphism onto $\text{im } f \subseteq N$ where the latter set is given the subspace topology.

Definition 11.24. Let S be an embedded *submanifold* of M , and let $\iota: S \hookrightarrow M$ denote the inclusion. A *retraction* from M to S is a map $r: M \rightarrow S$ such that $r \circ \iota = \text{id}_S$. A *deformation retraction* from M to S is a map $F: [0, 1] \times M \rightarrow M$ such that:

- (1) $F(0, -) = \text{id}_M$,

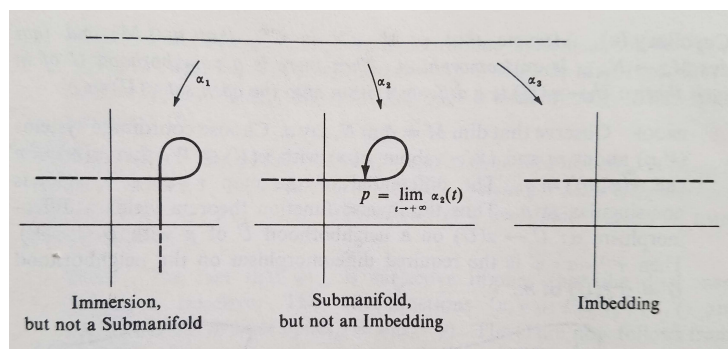


FIGURE 20. The difference between an immersion, a submanifold, and an embedding ([13]).

- (2) there is a retraction $r: M \rightarrow S$ such that $F(1, -) = \iota \circ r$, and
- (3) for all $t \in [0, 1]$, $F(t, -)|_S = \text{id}_S$.

Exercise 11.25. Show that a deformation retraction defines a homotopy equivalence.

Example 11.26. Let M be the open Möbius strip as in Example 6.5. Note that there is a deformation retraction from M to the unit circle S^1 (Figure 21). So they

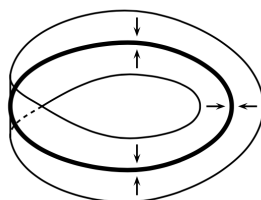


FIGURE 21. The Möbius band deformation retracts to its central circle.

have the same cohomology groups. Similarly, the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ has a deformation retraction to S^1 (Figure 22). Therefore, one can conclude that

$$H^k M = H^k(\mathbb{R}^2 \setminus \{(0, 0)\}) = H^k S^1.$$

Using the Mayer-Vietoris sequence, introduced in the next section, one may show that

$$H^k S^1 = \begin{cases} \mathbb{R} & \text{if } k = 0, 1; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 11.27. Let M be an orientable, closed n -dimensional manifold. Notice that the linear map

$$\int_M : H^n M \longrightarrow \mathbb{R}$$

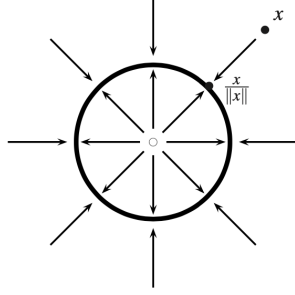


FIGURE 22. The punctured plane deformation retracts to the unit circle.

$$[\omega] \mapsto \int_M \omega$$

is well-defined by Stokes theorem:

$$\int_M (\omega + d\eta) = \int_M \omega + \int_M d\eta = \int_M \omega + \int_{\partial M} \eta = \int_M \omega$$

since $\partial M = \emptyset$.

Corollary 11.28. *Let M be an orientable, closed n -dimensional manifold and let $f: M \rightarrow N$ be smooth. Then the composition*

$$H^n N \xrightarrow{f^*} H^n M \xrightarrow{\int_M} \mathbb{R}$$

is homotopy invariant, i.e., if $f, g: M \rightarrow N$ are homotopic, then $\int_M \circ f^ = \int_M \circ g^*$.*

As an application of the homotopy invariance of de Rham cohomology, we prove the hairy ball theorem⁹, stated as follows:

Theorem 11.29. *Every vector field on an even-dimensional sphere has at least one zero.*

Proof. Let v be a nowhere vanishing vector field on the n -sphere S^n with n even. In other words, v is a section of tangent bundle, $v: S^n \rightarrow TS^n$ such that $v(x) \neq 0$ for all $x \in S^n$. Let D^{n+1} denote the solid unit ball in $\mathbb{R}^{n+1}$ with boundary $\partial D^{n+1} = S^n$. Now consider the antipodal involution (i.e., its square is the identity) on S^n ,

$$\tau: S^n \longrightarrow S^n$$

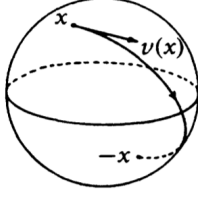
$$x = (x_1, \dots, x_{n+1}) \mapsto -x = (-x_1, \dots, -x_{n+1}).$$

We can think of $v(x)$ as the pointer towards $-x$ (see Figure 23) and construct a homotopy $\text{id}_{S^n} \stackrel{h}{\sim} \tau$:

$$h(t, x) = \cos(t\pi)x + \sin(t\pi) \frac{v(x)}{|v(x)|}.$$

Here, we are identifying $T_x S^n$ with vectors in $\mathbb{R}^{n+1}$ perpendicular to $x \in \mathbb{R}^n$ (thinking of tangent vectors in terms of curves on S^n passing through x). One can check that $\text{im}(h) \subset S^n$, that $h(0, -) = \text{id}_{S^n}$, and that $h(1, -) = \tau$.

⁹I did not make up the name of this theorem.

FIGURE 23. The vector $v(x)$ as a pointer.

Now by Corollary 11.28, for any n -form ω on S^n , we have

$$\int_{S^n} \omega = \int_{S^n} \tau^* \omega.$$

At the same time, note that τ is an orientation-reversing diffeomorphism when n is even ($\det(J\tau) = -1$ if and only if n is even). So we have

$$-\int_{S^n} \omega = \int_{S^n} \tau^* \omega,$$

forcing

$$\int_{S^n} \omega = 0.$$

However, since there are n -forms ω such that $\int_{S^n} \omega$ (bump functions, for example, or use that fact that S^n is orientable and hence has a nonvanishing n -form), we have a contradiction. $\square$

Note that S^1 , which is odd-dimensional, clearly has a nonvanishing vector field.

11.3. The Mayer-Vietoris sequence. We end this section with a long exact sequence that is a standard tool for computing cohomology.¹⁰

Let M be a smooth manifold and let $U, V \subseteq M$ be open. Define the inclusion maps i_U, i_V, j_U, j_V as shown in the following commutative diagram:

$$(7) \quad \begin{array}{ccccc} & & U & & \\ & j_U \nearrow & & \nwarrow i_U & \\ U \cap V & & & & U \cup V \\ & j_V \searrow & & \nearrow i_V & \\ & & V & & \end{array}$$

We want to use these inclusion maps to define cochain maps i and j such that the following sequence of cochain complexes is exact:

$$0 \rightarrow \Omega^\bullet(U \cup V) \xrightarrow{i} \Omega^\bullet U \oplus \Omega^\bullet V \xrightarrow{j} \Omega^\bullet(U \cap V) \rightarrow 0,$$

where the boundary map of $\Omega^\bullet U \oplus \Omega^\bullet V$ is (d, d) . This short exact sequence will then induce a long exact sequence called the *Mayer-Vietoris sequence*:

$$\cdots \xrightarrow{\partial} H^k(U \cup V) \xrightarrow{i} H^k U \oplus H^k V \xrightarrow{j} H^k(U \cap V) \xrightarrow{\partial} H^{k+1}(U \cup V) \xrightarrow{i} \cdots,$$

¹⁰For background on exact sequences, see Appendix D.

To see how this sequence would be useful in computing cohomology, suppose $M = U \cup V$ and that it is easy to compute the cohomology of U , V , and $U \cap V$. Then the cohomology groups of M are sandwiched in an exact sequence with known groups, which yields a lot of information. For instance, suppose that $H^{k-1}(U \cap V) = H^k(U) = H^k(V) = 0$. In that case, from the Mayer-Vietoris sequence, we know the following sequence is exact:

$$0 \rightarrow H^k M \rightarrow 0,$$

which tells us that $H^k M = 0$, too.

Lemma 11.30. *Using the notation above, there is an exact sequence of cochain complexes*

$$0 \rightarrow \Omega^\bullet(U \cup V) \xrightarrow{i} \Omega^\bullet U \oplus \Omega^\bullet V \xrightarrow{j} \Omega^\bullet(U \cap V) \rightarrow 0,$$

where i and j are defined to be

$$\begin{aligned} i(\omega) &= (i_U^* \omega, i_V^* \omega), \\ j(\omega_U, \omega_V) &= j_U^* \omega_U - j_V^* \omega_V. \end{aligned}$$

Proof. That i and j are cochain maps is easy to check, as is injectivity of i . We need to show, at each level k , that $\text{im}(i) = \ker(j)$ and that j is surjective.

We first show that $\text{im}(i) = \ker(j)$. Let $\omega \in \Omega^k(U \cup V)$. Since taking cohomology is functorial, it preserves the commutative diagram in (7), which implies

$$(j \circ i)(\omega) = j(i_U^* \omega, i_V^* \omega) = j_U^* i_U^* \omega - j_V^* i_V^* \omega = 0.$$

So $\text{im}(i) \subseteq \ker(j)$. Let $\omega_U \in \Omega^k U$ and $\omega_V \in \Omega^k V$. If $j(\omega_U, \omega_V) = j_U^* \omega_U - j_V^* \omega_V = 0$, then $\omega_U|_{U \cap V} = \omega_V|_{U \cap V}$. So we can glue ω_U and ω_V together along $U \cap V$ and obtain a form $\omega \in \Omega^k(U \cup V)$ and hence $(\omega_U, \omega_V) = i(\omega)$.

To see that j is surjective, let $\omega \in \Omega^k(U \cap V)$. Let $\{\lambda_U, \lambda_V\}$ be a partition of unity on $U \cup V$ subordinate to the cover $\{U, V\}$, i.e., $\lambda_U, \lambda_V: U \cup V \rightarrow [0, 1]$ are smooth and compactly supported with $\text{supp}(\lambda_U) \subseteq U$ and $\text{supp}(\lambda_V) \subseteq V$. Define

$$\omega_U = \lambda_U \omega, \quad \omega_V = \lambda_V \omega.$$

By defining $\omega_U(p) = 0$ outside of the support of λ_U , we may consider ω_U to be a form defined on all of U and with the property that $j_U^* \omega_U = \lambda_U \omega$ on $U \cap V$. A similar remark holds for ω_V . It follows that

$$\begin{aligned} j(\omega_U, -\omega_V) &= j_U^* \omega_U + j_V^* \omega_V \\ &= \lambda_U \omega + \lambda_V \omega \\ &= \omega, \end{aligned}$$

and we are done. $\square$

Theorem 11.31 (Mayer-Vietoris). *Let $U, V \subseteq M$ be open. Then there is a long exact sequence*

$$\cdots \longrightarrow H^k(U \cup V) \longrightarrow H^k U \oplus H^k V \longrightarrow H^k(U \cap V) \longrightarrow H^{k+1}(U \cup V) \longrightarrow \cdots.$$

Proof. The result follows from the snake lemma. Imitate the proof of Theorem D.3, which gives the long exact sequence in the dual setting of chain complexes. $\square$

Example 11.32. Consider $M = S^1 \subseteq \mathbb{R}^2$. We cover it with two open half circles U and V (overlapping a little bit) as shown in Figure 24. The Mayer-Vietoris sequence

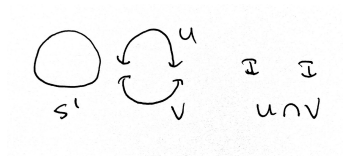


FIGURE 24. Cover S^1 with two open half circles.

is:

$$0 \rightarrow H^0 S^1 \rightarrow H^0 U \oplus H^0 V \rightarrow H^0(U \cap V) \rightarrow H^1 S^1 \rightarrow H^1 U \oplus H^1 V \rightarrow H^1(U \cap V) \rightarrow 0.$$

Notice that each of U and V is diffeomorphic to an open interval of $\mathbb{R}$ which is contractible. So $H^0 U = H^0 V = \mathbb{R}$ and $H^1 U = H^1 V = 0$. At the same time, the intersection $U \cap V$ can be contracted to two points, so $H^0(U \cap V) = \mathbb{R}^2$ and $H^1(U \cap V) = 0$. We already know that S^1 is connected, so $H^0 S^1 = \mathbb{R}$. The sequence then becomes

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2 \rightarrow H^1 S^1 \rightarrow 0.$$

Since the sequence is exact, the alternating sum of the dimensions of terms is 0 (via rank-nullity). We conclude that $\dim H^1 S^1 = 1$, and, hence, $H^1 S^1 = \mathbb{R}$.

Next, consider the two-sphere. We cover S^2 by two open half spheres as shown in Figure 25. Both U and V are homotopy equivalent to a point. Their intersection is

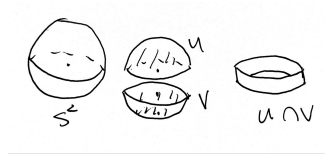


FIGURE 25. Cover S^2 with two open half spheres

the cylinder, which is homotopic equivalent to S^1 . So the Mayer-Vietoris sequence is:

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R} \rightarrow H^1 S^2 \rightarrow 0 \rightarrow \mathbb{R} \rightarrow H^2 S^2 \rightarrow 0.$$

We can break this into two exact sequence and conclude that

$$H^k S^2 = \begin{cases} \mathbb{R} & \text{if } k = 0, 2; \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 11.33. Generalize Example 11.32 by proving that the cohomology groups of the n -sphere is

$$H^k S^n = \begin{cases} \mathbb{R} & \text{if } k = 0, n; \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 11.34. Compute the cohomology of the twice punctured plane,

$$\mathbb{R}^2 \setminus \{(-1, 0), (1, 0)\}.$$

What about the plane with n holes?

Example 11.35. Let T denote the torus and consider a cover $T = U \cup V$ by two open cylinders as shown in Figure 26. Use A and B to denote the two connected components of $U \cap V$. Note that U , V , A , and B can all be contracted to S^1 .

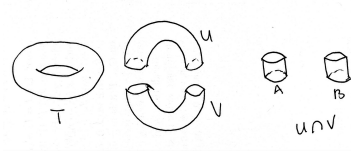


FIGURE 26. Cover the torus with two open cylinders.

So $U \cap V$ is homotopy equivalent to two circles and $H^2(U \cap V) = 0$ (since the union of two circles is a one-dimensional manifold). One can then use the cover $U \cap V = A \cup B$ and Mayer-Vietoris to show that $H^1(U \cap V) \cong H^1 A \oplus H^1 B = \mathbb{R}^2$ (or use the fact that the cohomology of a manifold is the direct sum of the cohomology of its components). The Mayer-Vietoris sequence becomes

$$0 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2 \rightarrow H^1 T \rightarrow \mathbb{R}^2 \xrightarrow{j} \mathbb{R}^2 \xrightarrow{\partial} H^2 T \rightarrow 0,$$

where ∂ is the connecting homomorphism, and $j: H^1 U \oplus H^1 V \rightarrow H^1(U \cap V)$ is induced by the map described in Lemma 11.30. By the exactness of the sequence, we have

$$H^2 T = \text{im}(\partial) \cong \mathbb{R}^2 / \ker(\partial) = \mathbb{R}^2 / \text{im}(j).$$

To find $\text{im}(j)$, note that an element $(a, b) \in H^1 U \oplus H^1 V$ is mapped to $(a - b, a - b) \in H^1(U \cap V)$ under j . So $\text{im}(j) \cong \mathbb{R}$ and $H^2 T \cong \mathbb{R}$. A dimension count then gives us that $H^1 T \cong \mathbb{R}^2$.

Note: Roughly, a torus has two independent one-dimensional holes and one two-dimensional hole.

12. DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS

A *Riemannian manifold* is a manifold with a metric. The metric allows us to talk about geometric properties of the manifold, for example, distances and angles. [some introduction](#)

12.1. Scalar products. Let V be a finite dimensional vector space over $\mathbb{R}$ with $\dim(V) = n$.

Definition 12.1. A symmetric bilinear form $\langle -, - \rangle: V \times V \rightarrow \mathbb{R}$ is *nondegenerate* if the map

$$\begin{aligned} \flat: V &\longrightarrow V^* \\ v &\longmapsto \langle v, - \rangle \end{aligned}$$

is an isomorphism. A nondegenerate symmetric bilinear form is called a *scalar product* on V .

Proposition 12.2. Let $v_1, \dots, v_n$ be a basis for V and suppose that $\langle -, - \rangle$ is a symmetric bilinear form on V . Let G be the $n \times n$ -matrix defined by

$$G_{i,j} = \langle v_i, v_j \rangle.$$

Then

(1) If $u = \sum_{i=1}^n a_i v_i$ and $w = \sum_{i=1}^n b_i v_i$, then

$$\langle u, v \rangle = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} G \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix};$$

(2) The form $\langle -, - \rangle$ is nondegenerate if and only if $\det(G) \neq 0$.

Exercise 12.3. Prove the above proposition.

Definition 12.4. Let $\langle -, - \rangle$ be a scalar product on V . A basis $v_1, \dots, v_n$ of V is called an *orthonormal basis* if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j; \\ \pm 1 & \text{if } i = j. \end{cases}$$

Remark 12.5. Given a scalar product, we can always find an orthonormal basis with respect to that scalar product (recall Gram-Schmidt).

Example 12.6. The Euclidean space $\mathbb{R}^n$ with the usual inner product has an orthonormal basis $e_1, \dots, e_n$.

Remark 12.7. Let $\langle -, - \rangle$ be a scalar product on V and let $e_1, \dots, e_n$ be an orthonormal basis. There is a scalar product on V^* induced by the product on V :

$$\langle e_i^*, e_j^* \rangle = \langle e_i, e_j \rangle,$$

and the matrices corresponding to the two scalar products with respect to the two bases are the same.

Remark 12.8. Let $\langle -, - \rangle$ be a scalar product on V . By the universal property of symmetric products, we have a unique map $s: \text{Sym}^2 V \rightarrow \mathbb{R}$ making the following diagram commute:

$$\begin{array}{ccc} V \times V & & \\ \downarrow & \searrow \langle -, - \rangle & \\ \text{Sym}^2 V & \xrightarrow[\exists!]{--} & \mathbb{R} \end{array}$$

Also, recall the isomorphism $(\text{Sym}^2 V)^* \cong \text{Sym}^2 V^*$ in Proposition 5.15 given by

$$\text{Sym}^l V^* \longrightarrow (\text{Sym}^l V)^*$$

$$\phi_1 \cdots \phi_l \longmapsto [v_1 \cdots v_l \mapsto \frac{1}{l!} \sum_{\sigma \in S_n} \prod_{i=1}^l \phi_{\sigma(i)}(v_i)].$$

Therefore, a scalar product on V is an element of $\text{Sym}^2 V^*$.

Example 12.9. Let M be a manifold of dimension 4 and let $p \in M$. Suppose that $T_p M$ has a basis e_1, e_2, e_3, e_4 . Consider $f = e_1^{*2} + e_2^{*2} + e_3^{*2} - e_4^{*2} \in \text{Sym}^2 T_p^* M$. It corresponds to a symmetric bilinear form on $T_p M$ defined to be

$$\begin{aligned} f(e_i, e_j) &= e_1^*(e_i)e_1^*(e_j) + e_2^*(e_i)e_2^*(e_j) + e_3^*(e_i)e_3^*(e_j) - e_4^*(e_i)e_4^*(e_j) \\ &= \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j \neq 4; \\ -1 & \text{if } i = j = 4, \end{cases} \end{aligned}$$

with the corresponding matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since G is nonsingular, f is a scalar product on $T_p M$.

Proposition 12.10. *Suppose that $\langle -, - \rangle$ is a scalar product on V . Let $e_1, \dots, e_n$ be a basis of V . Then the composition*

$$\begin{aligned} I: \Lambda^k V &\xrightarrow{\Lambda^k \flat} \Lambda^k V^* \xrightarrow{\cong} (\Lambda^k V)^* \\ e_\mu &\longmapsto \langle e_{\mu_1}, - \rangle \wedge \cdots \wedge \langle e_{\mu_n}, - \rangle \longmapsto [e_\nu \mapsto \det(\langle e_{\mu_i}, e_{\nu_j} \rangle)] \end{aligned}$$

induces a scalar product on $\Lambda^k V$.

Proof. Bilinear: Straightforward.

Symmetric: Note that

$$I(e_\mu)(e_\nu) = \det(\langle e_{\mu_i}, e_{\nu_j} \rangle) = \det(\langle e_{\nu_j}, e_{\mu_i} \rangle) = I(e_\nu)(e_\mu).$$

Nondegenerate: Now suppose that $e_1, \dots, e_n$ is an orthonormal basis of V . Observe that

$$I(e_\mu)(e_\nu) = \det(\langle e_{\mu_i}, e_{\nu_j} \rangle) = \begin{cases} 0 & \text{if } \mu \neq \nu; \\ \prod_{i=1}^k \langle e_{\mu_i}, e_{\nu_i} \rangle & \text{if } \mu = \nu \end{cases} = \pm 1.$$

So the matrix corresponding to I with respect to the basis $\{e_\mu\}$ is a diagonal matrix with diagonal entries being ± 1 . Hence I is nondegenerate. $\square$

Theorem 12.11 (Sylvester's Law of Intertia). *Let $\langle -, - \rangle$ be a scalar product on V . Then there exists a basis for V such that the matrix for V with respect to this basis is diagonal of the form*

$$G = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & -I_s \end{array} \right),$$

where I_k is the identity matrix of size k . The integer s is called the index of $\langle -, - \rangle$ and is independent of the choice of basis.

Proof. ? $\square$

Definition 12.12. A scalar product on V is *positive definite* if its index is 0.

12.2. The star operator. Now let V be a finite dimensional oriented vector space over $\mathbb{R}$ and let $\langle -, - \rangle$ be a scalar product on V .

Definition 12.13. Suppose that $e_1, \dots, e_n$ is a positively oriented orthonormal basis for V . The *volumn form* of V is the n -form over V^* :

$$\omega_V := e_1^* \wedge \dots \wedge e_n^*.$$

Lemma 12.14. *Let $e_1, \dots, e_n$ and $v_1, \dots, v_n$ be orthonormal bases for V . Suppose that $v_j = \sum_{i=1}^n a_{i,j} e_i$ and $v_j^* = \sum_{i=1}^n b_{i,j} e_i^*$. Define matrices A and B with $A_{i,j} = a_{i,j}$ and $B_{i,j} = b_{i,j}$. Then*

$$B = (A^\top)^{-1}.$$

Exercise 12.15. Prove Lemma 12.14.

Proposition 12.16. *The volumn form of V is the unique n -form sending each positively oriented orthonormal basis to 1. More generally, let $v_1, \dots, v_n$ be a positively oriented basis for V and let G be the matrix with $G_{i,j} = \langle v_i, v_j \rangle$. Then*

$$\omega_V = \sqrt{\det(G)} v_1^* \wedge \dots \wedge v_n^*,$$

and in particular, if $v_1, \dots, v_n$ is orthonormal, then $\omega = v_1^* \wedge \dots \wedge v_n^*$.

Proof. Let $e_1, \dots, e_n$ be a positively oriented orthonormal basis for V . Let $\tilde{I}$ be the matrix for $\flat: V \rightarrow V^*$ with respect to the basis $e_1, \dots, e_n$, i.e., $\tilde{I} = \text{diag}(\epsilon_1, \dots, \epsilon_n)$, where $\epsilon_i = \langle e_i, e_i \rangle \in \{\pm 1\}$. Suppose that $v_j = \sum_{i=1}^n a_{i,j} e_i$. Then we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{A} & V \\ G \downarrow & & \downarrow \tilde{I} \\ V^* & \xleftarrow{A^\top} & V^* \end{array}$$

i.e., $G = A^\top \tilde{I} A$. So $\det(G) = \det(A)^2 \det(\tilde{I})$. Since both $v_1, \dots, v_n$ and $e_1, \dots, e_n$ are positively oriented, $\det(A) > 0$. Hence,

$$\det(A) = \sqrt{|\det(G)|}.$$

Finally, by Lemma 12.14, we have

$$v_1^* \wedge \cdots \wedge v_n^* = \det((A^\top)^{-1}) e_1^* \wedge \cdots \wedge e_n^*,$$

which is the same as

$$e_1^* \wedge \cdots \wedge e_n^* = \sqrt{|\det(G)|} v_1^* \wedge \cdots \wedge v_n^*.$$

□

Theorem 12.17 (The star operator). *Let V be a finite-dimensional oriented vector space over $\mathbb{R}$ with a scalar product $\langle -, - \rangle$ and let ω_V be its volume form. Then for each $k \geq 0$, there exists a unique linear map*

$$*: \Lambda^k V^* \rightarrow \Lambda^{n-k} V^*$$

satisfying

$$\eta \wedge (*\zeta) = \langle \eta, \zeta \rangle \omega_V$$

for all $\eta, \zeta \in \Lambda^k V^*$. We call the linear mapping the Hodge $*$ -operator, or the star operator.

Proof. Let $e_1, \dots, e_n$ be an orthonormal basis for V and use $\mathcal{B} = \{e_\mu^*\}_{1 \leq \mu_1 < \dots < \mu_k \leq n}$ to denote the corresponding basis for $\Lambda^k V^*$. Since $*$ is determined by its action on the basis vectors, without loss of generality, suppose that $\zeta = e_\mu^*$ and write $*\zeta = \sum_\nu a_\nu e_\nu^* \in \Lambda^{n-k} V^*$. We want to compute the coefficients $\{a_\nu\}_\nu$ such that

$$\eta \wedge (*\zeta) = \langle \eta, \zeta \rangle \omega_V$$

for all $\eta \in \Lambda^k V^*$. Again, we only need to consider the case $\eta = e_\gamma^* \in \mathcal{B}$. Note that

$$\langle \eta, \zeta \rangle = \langle e_\gamma^*, e_\mu^* \rangle = \det(\langle e_{\gamma_i}, e_{\mu_j} \rangle) = \begin{cases} 0 & \text{if } \gamma \neq \mu; \\ \epsilon_\mu & \text{if } \gamma = \mu, \end{cases}$$

where $\epsilon_\mu = \prod_{i=1}^k \langle e_{\gamma_i}, e_{\mu_i} \rangle$. So $\langle \eta, \zeta \rangle \in \{\pm 1\}$ if $\gamma = \mu$. On the other hand, by definition we have

$$\eta \wedge *\zeta = a_{\tilde{\gamma}} e_\gamma^* \wedge e_{\tilde{\gamma}}^*,$$

where $\tilde{\gamma}$ is the index formed by $[n] \setminus \{\gamma_1, \dots, \gamma_k\}$ arranged in increasing order. Let τ_γ be the permutation sending $(1, \dots, n)$ to $(\gamma_1, \dots, \gamma_k, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{n-k})$. We have

$$\eta \wedge *\zeta = a_{\tilde{\gamma}} \text{sign}(\tau_\gamma) \omega_V.$$

Therefore, by assumption,

$$a_{\tilde{\gamma}} \text{sign}(\tau_\gamma) = \langle \eta, \zeta \rangle = \begin{cases} 0 & \text{if } \gamma \neq \mu; \\ \epsilon_\mu & \text{if } \gamma = \mu. \end{cases}$$

Hence,

$$a_{\tilde{\gamma}} = \text{sign}(\tau_\gamma) \langle \eta, \zeta \rangle = \begin{cases} 0 & \text{if } \gamma \neq \mu; \\ \text{sign}(\tau_\gamma) \epsilon_\mu & \text{if } \gamma = \mu. \end{cases}$$

So we define $*\zeta = *e_\mu^*$ to be

$$*e_\mu^* := \text{sign}(\tau_\mu) \epsilon_\mu e_\mu^*.$$

□

Remark 12.18. In summary, the star operator is defined to be

$$\begin{aligned} *: \Lambda^k V^* &\longrightarrow \Lambda^{n-k} V^* \\ e_\mu^* &\longmapsto \text{sign}(\tau_\mu) \epsilon_\mu e_{\tilde{\mu}}^*, \end{aligned}$$

where $\epsilon_\mu = \prod_{i=1}^k \langle e_{\mu_i}, e_{\mu_i} \rangle = \pm 1$, $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_{n-k})$ with $\tilde{\mu}_1 < \dots < \tilde{\mu}_{n-k}$ and $\{\tilde{\mu}_1, \dots, \tilde{\mu}_{n-k}\} = [n] \setminus \{\mu_1, \dots, \mu_k\}$, and τ_μ is the permutation that sends $(1, \dots, n)$ to $(\mu_1, \dots, \mu_k, \tilde{\mu}_1, \dots, \tilde{\mu}_{n-k})$. Hence, $e_\mu^* \wedge e_{\tilde{\mu}}^* = \text{sign}(\tau_\mu) \omega_V$.

Example 12.19. Consider $\mathbb{R}^3$ with the usual scalar product and let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$. Then

$$*(e_1^* \wedge e_2^*) = e_3^*, \quad *(e_1^* \wedge e_3^*) = -e_2^*, \quad *(e_2^* \wedge e_3^*) = e_1^*.$$

Example 12.20. Consider $\mathbb{R}^4$ with the scalar product of index 1 and let $e_1, \dots, e_4$ be an orthonormal basis. Let $G = \text{diag}(1, 1, 1, -1)$ be the corresponding matrix for this scalar product with respect to the basis. Then

$$*(e_1^* \wedge e_3^*) = \pm e_2^* \wedge e_4^*.$$

We have $\mu = (1, 3)$ and we want to find the sign. Note that $\tau_\mu = (23)$, so $\text{sign}(\tau_\mu) = -1$. Also, $\epsilon_\mu = \langle e_1, e_1 \rangle \cdot \langle e_3, e_3 \rangle = 1$. Therefore,

$$*(e_1^* \wedge e_3^*) = -e_2^* \wedge e_4^*.$$

For another example,

$$*(e_1^* \wedge e_2^* \wedge e_4^*) = \text{sign}(34) \cdot 1 \cdot 1 \cdot (-1) e_3^* = e_3^*.$$

Lemma 12.21. Let V be an oriented vector space over $\mathbb{R}$ and let $\langle -, - \rangle$ be a scalar product of index s . For each k ,

$$** = (-1)^{k(n-k)+s} \text{id}_{\Lambda^k V^*}.$$

Hence, $*$ is an isomorphism.

Exercise 12.22. Prove Lemma 12.21.

Exercise 12.23. Let V be an oriented vector space over $\mathbb{R}$ and let $\langle -, - \rangle$ be a scalar product of index s . Show that for all $\eta, \zeta \in \Lambda^k V^*$

$$\langle *\eta, *\zeta \rangle = (-1)^s \langle \eta, \zeta \rangle.$$

12.3. Poincaré duality.

Definition 12.24. A *semi-Riemannian manifold of index s* is a manifold M together with a family

$$\langle -, - \rangle = \{\langle -, - \rangle_p\}_{p \in M}$$

of scalar products $\langle -, - \rangle_p$ of index s on $T_p M$ for all $p \in M$ varying smoothly with p (i.e., the entries in the corresponding matrix are smooth locally around p .) If all the forms are positive definite, i.e., $s = 0$, then we call $(M, \langle -, - \rangle)$ a *Riemannian manifold*.

Remark 12.25. The collection of scalar products $\langle -, - \rangle_p$ of a Riemannian manifold is sometimes called a *Riemannian metric*. Note that every oriented manifold can be given such a metric using the usual scalar product on $\mathbb{R}^n$ and a partition of unity.

Remark 12.26. We saw in Remark 12.8 that a scalar product on a vector space V corresponds with an element of the symmetric product $\text{Sym}^2 V^*$. Thus, we can define a semi-Riemannian manifold as a manifold together with a section of the bundle $\text{Sym}^2 T^*M$ (the fiber at each $p \in M$ is $\text{Sym}^2 T_p^*M$) such that the corresponding bilinear form on T_pM is nondegenerate for all $p \in M$.

Suppose that M is an oriented semi-Riemannian n -manifold of index s . For each $p \in M$, on the tangent space T_pM we get a volume form ω_{T_pM} and a star operator $*$: $\Lambda^k T_p^*M \rightarrow \Lambda^{n-k} T_p^*M$ that vary smoothly with p . Gluing them together, we obtain a *volume form* $\omega_M \in \Omega^n M$ on M and a *star operator* $*$: $\Omega^k M \rightarrow \Omega^{n-k} M$ on M .

Definition 12.27. The *coderivative* $\delta_k: \Omega^{n-k} M \rightarrow \Omega^{n-k-1} M$ is defined by

$$\delta_k := (-1)^k * d *^{-1}.$$

The coderivative allows us to have the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Omega^0 M & \xrightarrow{d} & \Omega^1 M & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1} M & \xrightarrow{d} & \Omega^n M & \longrightarrow & 0 \\ & & \cong \downarrow * & & \cong \downarrow * & & & & \cong \downarrow * & & \cong \downarrow * & & \\ 0 & \longrightarrow & \Omega^n M & \xrightarrow{\delta} & \Omega^{n-1} M & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & \Omega^1 M & \xrightarrow{\delta} & \Omega^0 M & \longrightarrow & 0 \end{array}$$

Now let M be a closed Riemannian n -manifold. We will show that $*$ induces an isomorphism of cohomology groups $H^k M \cong H^{n-k} M$, which gives the Poincaré duality. By Lemma 12.21, for all $\zeta \in \Omega^{k+1} M$ we have

$$d * \zeta = (-1)^{k(n-k)} * d * *^{-1} \zeta = (-1)^{k(n-k)+(k+1)(n-k-1)} * d *^{-1} \zeta = (-1)^k * \delta \zeta.$$

Exercise 12.28. Let $\eta \in \Omega^k M$ and $\zeta \in \Omega^{k+1} M$. Recall the product rule

$$d(\eta \wedge * \zeta) = d\eta \wedge * \zeta + (-1)^k \eta \wedge d * \zeta.$$

Show that

$$d(\eta \wedge * \zeta) = d\eta \wedge * \zeta + \eta \wedge * \delta \zeta.$$

Remark 12.29. The coderivative δ is independent of the orientation of M . We define a positive-definite scalar product on $\Omega^k M$ using the scalar product on each of $T_p M$ as follows: For all $\eta, \zeta \in \Omega^k M$, put

$$\langle \langle \eta, \zeta \rangle \rangle := \int_M \langle \eta, \zeta \rangle \omega_M = \int_M \eta \wedge * \zeta.$$

Since by assumption M is closed, by Stokes' theorem, $\int_M d(\eta \wedge * \zeta) = 0$. So the above exercise shows that for all $\eta \in \Omega^k M$ and $\zeta \in \Omega^{k+1} M$,

$$\langle \langle d\eta, \zeta \rangle \rangle + \langle \langle \eta, \delta \zeta \rangle \rangle = 0.$$

This means that $-\delta$ is the *adjoint* of d , i.e.,

$$\langle\langle d\eta, \zeta \rangle\rangle = \langle\langle \eta, -\delta\zeta \rangle\rangle.$$

At the same time, the scalar product on $\Omega^k M$ gives us for free that

$$\ker(d_k) = (\operatorname{im}(\delta_{n-k-1}))^\perp, \quad \ker(\delta_{n-k}) = (\operatorname{im}(d_{k-1}))^\perp.$$

Theorem 12.30. *Let M be an oriented closed Riemannian n -manifold. Then*

$$\Omega^k M = \ker(d_k) \oplus \operatorname{im}(\delta_{n-k-1}) = \ker(\delta_{n-k}) \oplus \operatorname{im}(d_{k-1}).$$

as an orthogonal direct sum with respect to the scalar product on $\Omega^k M$ defined by

$$\langle\langle \eta, \zeta \rangle\rangle := \int_M \eta \wedge * \zeta.$$

Remark 12.31. The proof is beyond the scope of these notes, but the interested reader can consult Chapter 6 of [13].

Corollary 12.32. *For M as above,*

$$\begin{aligned} \ker(d_k) &= \operatorname{im}(d_{k-1}) \oplus (\ker(d_k) \cap \ker(\delta_{n-k})), \\ \ker(\delta_{n-k}) &= \operatorname{im}(\delta_{n-k-1}) \oplus (\ker(d_k) \cap \ker(\delta_{n-k})). \end{aligned}$$

This decomposition led us to consider the *harmonic* forms on M .

Definition 12.33. The *harmonic k -forms* on M are

$$\mathcal{H}^k M = \{\eta \in \Omega^k M \mid d\eta = 0 \text{ and } \delta\eta = 0\}.$$

Theorem 12.34 (Hodge Theorem). *Every de Rham cohomology class of an oriented closed Riemannian manifold is represented by a well-defined harmonic form. More precisely, for each k , the canonical map*

$$\begin{aligned} \mathcal{H}^k M &\longrightarrow H^k M \\ \eta &\longmapsto [\eta] \end{aligned}$$

is an isomorphism.

Proof. This follows from Corollary 12.32. □

Remark 12.35. As a result, we have an orthonormal direct sum decomposition

$$\Omega^k M = d\Omega^{k-1} M \oplus \delta\Omega^{k+1} M \oplus \mathcal{H}^k M.$$

Proposition 12.36. *The star operator $*$: $\mathcal{H}^k \rightarrow \mathcal{H}^{n-k} M$ is an isomorphism.*

Proof. Let $\eta \in \mathcal{H}^k M$. So $d\eta = \delta\eta = 0$. Then

$$d * \eta = \pm \delta * \eta = 0, \quad \delta * \eta = \pm * d\eta = 0.$$

Hence, $*(\mathcal{H}^k M) \subseteq \mathcal{H}^{n-k} M$. It is an isomorphism since $** = \pm \operatorname{id}_M$. □

Theorem 12.37 (Poincaré Duality). *Let M be an oriented closed Riemannian n -manifold. Then $H^k M \cong H^{n-k} M$ as described in the following diagram:*

$$\begin{array}{ccc}
\mathcal{H}^k M & \xrightarrow{\cong} & H^k M \\
* \downarrow \cong & & \cong \downarrow \text{Poincaré} \\
\mathcal{H}^{n-k} M & \xrightarrow{\cong} & H^{n-k} M
\end{array}$$

Corollary 12.38. *Let M be a connected oriented closed Riemannian n -manifold. Then $H^n M \cong \mathbb{R}$.*

13. TORIC VARIETIES

An n -dimensional manifold over $\mathbb{C}$ is a second countable Hausdorff space where the charts are homeomorphic to open subsets of $\mathbb{C}^n$ and the transition maps are holomorphic (locally given by convergent power series). In this section we look at smooth *toric varieties* as examples of complex manifolds. The name “toric” comes from the action on the variety by the algebraic torus $(\mathbb{C}^*)^n$ (where $\mathbb{C}^*$ is the set of nonzero complex numbers, i.e., $\mathbb{R}^2 \setminus \{(0,0)\}$).

13.1. Toric varieties from fans. Let N be a *lattice* of rank n , meaning that it is a free $\mathbb{Z}$ -module generated by some basis $w_1, \dots, w_n \in N$. So we have $N \cong \mathbb{Z}^n$ as $\mathbb{Z}$ -modules. Consider the vector space over $\mathbb{R}$

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$$

obtained from extension of scalars. We can think of N as a collection of points in $N \otimes_{\mathbb{Z}} \mathbb{R}$ that have integer coordinates. Let $M = \text{hom}(N, \mathbb{Z})$ be the dual lattice and use $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ to denote the corresponding vector space. There is a *dual pairing* $\langle -, - \rangle: M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \langle -, - \rangle: M_{\mathbb{R}} \times N_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ (f, v) &\longmapsto f(v). \end{aligned}$$

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$ (hence a basis for $N_{\mathbb{R}}$). Then in coordinates, the dual pairing can be written as

$$\langle \sum_{i=1}^n a_i e_i^*, \sum_{i=1}^n b_i e_i \rangle = \sum_{i=1}^n a_i b_i.$$

Further identifying $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$, the pairing $M \times N$ is just the ordinary inner product.

A *convex polyhedral cone* is the set

$$\sigma = \{ \sum_{i=1}^s r_i v_i \mid r_i \geq 0 \}$$

for some $v_1, \dots, v_s \in N_{\mathbb{R}}$. The cone is *rational* if $v_i \in N$. It is *strongly convex* if it does not contain a line through the origin. The *dual cone* $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ is defined to be the set

$$\sigma^{\vee} := \{ u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}.$$

A *face* τ of σ is the intersection of σ with any supporting hyperplane: $\tau = \sigma \cap u^{\perp} = \{ v \in \sigma \mid \langle u, v \rangle = 0 \}$ for some $u \in \sigma^{\vee}$. A face of a cone is also a cone.

Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. There is a *semigroup* associated with σ :

$$S_{\sigma} := \sigma^{\vee} \cap M.$$

The *group algebra* corresponding to σ is

$$\mathbb{C}[S_{\sigma}] := \mathbb{C}[e_u \mid u \in S_{\sigma}]$$

whose elements are polynomials in the symbols e_u with the relations

$$e_u e_v := e_{u+v} \quad \text{and} \quad e_0 := 1.$$

It turns out that S_σ is finitely generated over $\mathbb{Z}_{\geq 0}$, i.e., $\mathbb{C}[S_\sigma]$ is finitely generated; that is, fixing $u_1, \dots, u_s$, a minimal set of generators for S_σ , we have

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[x_1, \dots, x_s]/I,$$

where I is the ideal generated by binomials corresponding to the relations between the u_i 's:

$$x_1^{a_1} \cdots x_n^{a_n} - x_1^{b_1} \cdots x_n^{b_n}$$

where $\sum_{i=1}^s a_i u_i = \sum_{i=1}^s b_i u_i$. By Hilbert's theorem, a finite number of these generators will suffice. Denote the *zero set* or *vanishing set* of I as

$$U_\sigma := Z(I) := \{p \in \mathbb{C}^s \mid f(p) = 0 \text{ for all } f \in I\}.$$

Alternatively, if we treat $u_i \in \mathbb{Z}^n$ as exponents for monomials in $x_1, \dots, x_n$, we have

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[x^{u_1}, \dots, x^{u_s}] \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where for $a \in \mathbb{Z}^n$, we use the notation $x^a := \prod_{i=1}^n x_i^{a_i}$.

Definition 13.1. Let σ be a strongly convex polyhedral cone in N . The *affine toric variety* associated to σ , denoted U_σ , is the zero set constructed as above.

Example 13.2. Let $n = 1$. Note that $\tau = \{0\}$ is a cone. Its dual cone is $\tau^\vee = M_{\mathbb{R}} \cong \mathbb{R}$, which is generated by $v_1 = e_1^* = 1$ and $v_2 = -e_1^* = -1$ with the single relation $v_1 + v_2 = 0$. So

$$\mathbb{C}[S_\tau] = \mathbb{C}[x_1^{v_1}, x_1^{v_2}] = \mathbb{C}[x_1, x_1^{-1}] \cong \mathbb{C}[x, y]/(xy - 1)$$

and the zero set of $xy - 1$ is

$$\begin{aligned} U_\tau &= \{(u, 1/u) \in \mathbb{C}^2 \mid u \neq 0\} \cong \mathbb{C}^* := \mathbb{C} \setminus \{0\} \\ &\quad (u, 1/u) \mapsto u \\ &\quad (z, 1/z) \leftarrow z. \end{aligned}$$

In general, for an arbitrary n and $\tau = \{0\}$, we have

$$\mathbb{C}[S_\tau] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cong \mathbb{C}[x_1, y_1, \dots, x_n, y_n]/(x_1 y_1 - 1, \dots, x_n y_n - 1)$$

and $U_\tau = (\mathbb{C}^*)^n$, the n -torus. For any cone σ , the affine toric variety U_σ turns out to contain the torus as a dense open subset.

If τ is a facet (i.e., a codimension one face) of σ , then there is an inclusion $U_\tau \subseteq U_\sigma$ which can be described as follows: Suppose that $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$. Without loss of generality, say $u = v_s$ is the last generator of σ . Then $S_\sigma \subseteq S_\tau$ since the generators of S_τ are the generators of S_σ and $-v_s$, and as a result, $\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[S_\tau]$ as a subalgebra. To be more explicit, the relations amongst the generators of S_τ are the relations amongst those of S_σ plus the additional relation $v_s + (-v_s) = 0$. This means that we have

$$\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[S_\tau] \cong \mathbb{C}[y_1, \dots, y_s, y_{s+1}]/(I + (y_s y_{s+1} - 1)),$$

where we recall that $\mathbb{C}[S_\sigma] \cong \mathbb{C}[y_1, \dots, y_s]/I$. Therefore, U_τ is subject to one more restriction than U_σ , and we can identify U_τ as

$$U_\tau \cong U_\sigma \setminus \{y_s = 0\},$$

i.e., points in U_σ that have nonzero entry in the last coordinate.

Example 13.3. Consider the case $n = 2$, and take σ to be generated by e_2 and $2e_1 - e_2$ as shown in the left of Figure 27. The cone in the right is the dual lattice $\sigma^\vee$ with the three generators of S_σ being circled. The three generators are

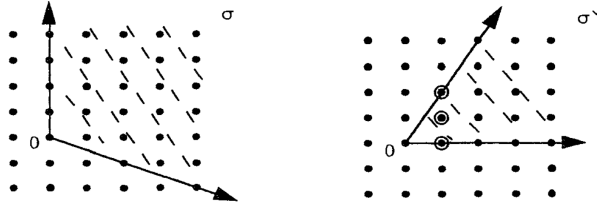


FIGURE 27. A cone σ and its dual. The generators of S_σ are circled.

$u_1 = e_1^* = (1, 0)$, $u_2 = e_1^* + e_2^* = (1, 1)$ and $u_3 = e_1^* + 2e_2^* = (1, 2)$. The corresponding group algebra is

$$(8) \quad \mathbb{C}[S_\sigma] = \mathbb{C}[x^{u_1}, x^{u_2}, x^{u_3}] = \mathbb{C}[x_1, x_1x_2, x_1x_2^2] \cong \mathbb{C}[y_1, y_2, y_3]/(y_2^2 - y_1y_3),$$

and $U_\sigma = \{(y_1, y_2, y_3) \in \mathbb{C}^3 \mid y_2^2 = y_1y_3\}$ is a conic surface whose real points are pictured in Figure 28. Now let $\tau = \mathbb{R}_{\geq 0}e_2$ be the cone generated by e_2 . Then $\tau^\vee$ is

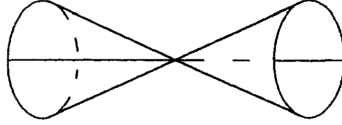


FIGURE 28. The real points of the affine toric variety U_σ in Example 13.3.

the upper half plane, and S_τ is generated by $e_1^*, -e_1^*, e_2^*$. We have

$$(9) \quad \mathbb{C}[S_\tau] = \mathbb{C}[x_1, x_1^{-1}, x_2] \cong \mathbb{C}[z_1, z_2, z_3]/(z_1z_2 - 1).$$

So

$$U_\tau \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C}$$

$$(z_1, z_2, z_3) \mapsto (z_1, z_3),$$

and the map $U_\tau \rightarrow U_\sigma$ can be described by

$$\mathbb{C}^* \times \mathbb{C} \longrightarrow U_\sigma \setminus \{(y_1, y_2, y_3) \in U_\sigma \mid y_1 \neq 0\}$$

$$(z_1, z_3) \longmapsto (z_1, z_1z_3, z_1z_3^2).$$

Here is a way to find the above mapping. Under the isomorphisms Equation 8 and Equation 9, we have

$$y_1 \leftrightarrow x_1, \quad y_2 \leftrightarrow x_1 x_2, \quad y_3 \leftrightarrow x_1 x_2^2$$

and

$$z_1 \leftrightarrow x_1, \quad z_2 \leftrightarrow x_1^{-1}, \quad z_3 \leftrightarrow x_2.$$

Eliminating the x_i , we find $z_1 = y_1$, $y_2 = z_1 z_3$, and $y_3 = z_1 z_3^2$.

A fan Δ in $N_{\mathbb{R}}$ is a collection of strongly convex rational polyhedral cones σ in $N_{\mathbb{R}}$ such that

- (1) If τ is a face of σ , then $\tau \in \Delta$;
- (2) If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau \in \Delta$.

Construction of a toric variety from a fan. Let Δ be a fan in $N_{\mathbb{R}}$. The *toric variety* associated to Δ , denoted $X(\Delta)$ is constructed by starting with the disjoint union of the affine toric varieties U_{σ} for all $\sigma \in \Delta$. We then glue these varieties together according to the following instructions. For each pair of cones $\sigma, \sigma' \in \Delta$ we have induced mappings

$$\mathbb{C}[S_{\sigma}] \hookrightarrow \mathbb{C}[S_{\sigma \cap \sigma'}] \hookrightarrow \mathbb{C}[S_{\sigma'}] \quad \rightsquigarrow \quad U_{\sigma} \hookrightarrow U_{\sigma \cap \sigma'} \hookrightarrow U_{\sigma'}.$$

Write

$$U_{\sigma} \xrightarrow{\phi_{\sigma, \sigma'}} U_{\sigma'}$$

for the mapping that sends the image of each point $p \in U_{\sigma \cap \sigma'}$ in U_{σ} to the image of p in $U_{\sigma'}$. The dashed arrow indicates that the mapping is not defined on the whole domain and codomain. (However, it is a bijection on the images of $U_{\sigma \cap \sigma'}$.) In this way, we glue U_{σ} to $U_{\sigma'}$ along $U_{\sigma \cap \sigma'}$ and $\phi_{\sigma, \sigma'}$. For each pair of cones of Δ , we perform these gluings to arrive at the toric variety $X(\Delta)$. We then have $U_{\sigma} \subseteq X(\Delta)$ for each $\sigma \in \Delta$, and the mappings $\phi_{\sigma, \sigma'}$ then become transition functions. We omit the verification that all of these gluings are compatible.

Definition 13.4. Let Δ be a fan in $N_{\mathbb{R}}$. The *toric variety* associated to Δ , denoted $X(\Delta)$, is constructed from Δ by gluing the affine toric varieties U_{σ} for all $\sigma \in \Delta$.

Example 13.5. Consider the fan Δ in $\mathbb{R}$ consisting of the three cones as shown in Figure 29. We have

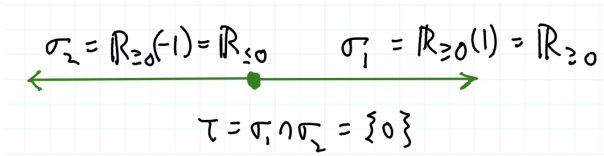


FIGURE 29. A fan Δ such that $X(\Delta) = \mathbb{P}^1$.

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[x], \quad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[y], \quad \mathbb{C}[S_{\tau}] = \mathbb{C}[x, y]/(xy - 1).$$

So $U_{\sigma_1} = U_{\sigma_2} = \mathbb{C}$, $U_{\tau} = \mathbb{C}^*$, and the gluing instruction is given by

$$\begin{aligned}\phi_{\sigma_1, \sigma_2}: U_{\sigma_1} &\dashrightarrow U_{\sigma_2} \\ x &\mapsto 1/x.\end{aligned}$$

Therefore, $X(\Delta) = \mathbb{P}^1$.

Example 13.6. Let $n = 2$. Consider the fan Δ with cones $\sigma_1, \sigma_2, \sigma_3$ and their faces, where σ_1 is generated by e_1, e_2 , σ_2 is generated by $e_2, -e_1 - e_2$, and σ_3 is generated by $e_1, -e_1 - e_2$ as shown on the left in Figure 30.

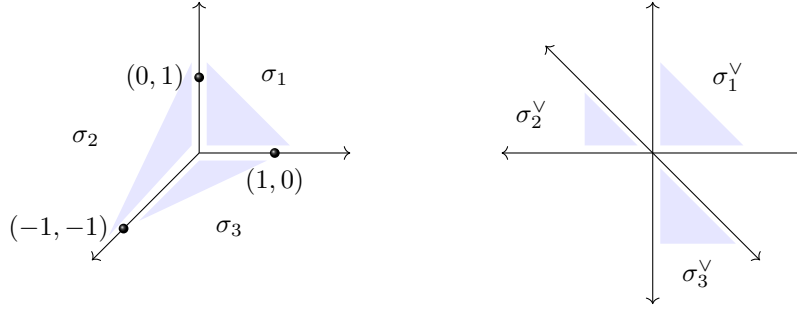


FIGURE 30. Fan Δ in $\mathbb{R}^2$ such that $X(\Delta) = \mathbb{P}^2$.

We claim that the corresponding toric variety is $\mathbb{P}^2$. Indeed, one can see that

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[u, v], \quad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[u^{-1}, u^{-1}v], \quad \mathbb{C}[S_{\sigma_3}] = \mathbb{C}[v^{-1}, vu^{-1}],$$

and as a result,

$$U_{\sigma_1} = U_{\sigma_2} = U_{\sigma_3} = \mathbb{C}^2.$$

However, notice that their intersections are $\mathbb{C}^* \times \mathbb{C}$ and the gluing maps are given by

$$\begin{aligned}U_{\sigma_1} &\dashrightarrow U_{\sigma_2} &\dashrightarrow U_{\sigma_3} \\ (u, v) &\mapsto (1/u, v/u) &\mapsto (1/v, v/u).\end{aligned}$$

Relabeling u and v as x_2/x_1 and x_3/x_1 , we see that the gluing morphisms are our usual transition maps of $\mathbb{P}^2$.

Exercise 13.7. Let $e_{n+1} = -e_1 - \dots - e_n$. Show that $\mathbb{P}^n$ can be constructed as a toric variety using the fan Δ with cones generated by the proper subsets of $e_1, \dots, e_{n+1}$. To start with, let σ_i be the cone over $e_1, \dots, \hat{e}_i, \dots, e_{n+1}$, a maximal cone in Δ . One can think of U_{σ_i} as the standard chart U_i of $\mathbb{P}^n$ and the gluing morphisms are exactly the usual transition functions. You might find [this package](#) of **SageMath** helpful when finding the generators of the dual cone $\sigma_i^\vee$.

Example 13.8 (Hirzebruch Surfaces). Consider the fan depicted in Figure 31, where the slanting arrow passes through the point $(-1, a)$ for some $a \in \mathbb{Z}_{>0}$. The corresponding toric variety is called the *Hirzebruch surface* and is denoted H_a . The reader should construct the dual cones and verify that

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[u, v], \quad \mathbb{C}[S_{\sigma_2}] = \mathbb{C}[u, v^{-1}],$$

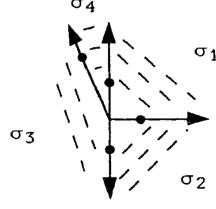


FIGURE 31. Fan for the the Hirzebruch surface.

$$\mathbb{C}[S_{\sigma_3}] = \mathbb{C}[u^{-1}, u^{-a}v^{-1}], \quad \mathbb{C}[S_{\sigma_4}] = \mathbb{C}[u^{-1}, u^av].$$

Gluing U_{σ_1} to U_{σ_2} and gluing U_{σ_3} to U_{σ_4} yields two copies of $\mathbb{C} \times \mathbb{P}^1$. The other two gluing maps are given by

$$\begin{array}{ccc} U_{\sigma_1} & \dashrightarrow & U_{\sigma_4} \\ (u, v) & \mapsto & (1/u, u^av), \end{array} \quad \begin{array}{ccc} U_{\sigma_2} & \dashrightarrow & U_{\sigma_3} \\ (x, y) & \mapsto & (1/x, y/x^a). \end{array}$$

We would now like to define a projection $H_a \rightarrow \mathbb{P}^1$ whose fibers are all $\mathbb{P}^1$, i.e., the mapping is surjective and the inverse image of each point in $\mathbb{P}^1$ is a copy of $\mathbb{P}^1$. We say H_a is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. To construct the projection, consider the mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the first projection $(x, y) \mapsto x$. It maps the lattice $N = \mathbb{Z}^2 \subset \mathbb{R}^2$ to the lattice $N' = \mathbb{Z} \subset \mathbb{R}$. It also maps each cone σ_1 and σ_2 to the cone $\tau_1 := \mathbb{R} \cdot 1$ and each cone σ_3 and σ_4 to the cone $\tau_2 := \mathbb{R} \cdot (-1)$. Let Δ' be the fan in $\mathbb{R}$ with maximal cones τ_1 and τ_2 . As we saw earlier, $X(\Delta') = \mathbb{P}^1$. In a way specified in general below, the mapping $N \rightarrow N'$ has induced a *mapping of fans* $\Delta \rightarrow \Delta'$, and in turn this induces a mapping of toric varieties

$$H_a = X(\Delta) \rightarrow X(\Delta') = \mathbb{P}^1.$$

We have the following commutative mapping in which the dashed arrows are gluing mappings

$$\begin{array}{ccccc} U_{\sigma_4} & & (1/u, u^av) & \dashleftarrow & (u, v) & & U_{\sigma_1} \\ & & \updownarrow & & \updownarrow & & \\ U_{\sigma_3} & & (1/u, 1/(u^av)) & \dashleftarrow & (u, 1/v) & & U_{\sigma_2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U_{\tau_2} & & 1/u & \dashleftarrow & u & & U_{\tau_1} \end{array}$$

These mappings glue together to give the projection $H_a \rightarrow \mathbb{P}^1$. Note that the inverse image of every point in $\mathbb{P}^1$ is a $\mathbb{P}^1$.

In general, suppose we have a $\mathbb{Z}$ -linear mapping $\phi: N \rightarrow N'$ of lattices and fans Δ and Δ' in N and N' , respectively, having the property that for each cone $\sigma \in \Delta$, there is a cone $\sigma' \in \Delta'$ such that $\phi(\sigma) \subseteq \sigma'$, then—just as for the example of the Hirzebruch surface, above—there is a natural mapping $X(\Delta) \rightarrow X(\Delta')$ induced by ϕ . In the example of the Hirzebruch surface we used $\phi(x, y) = x$. Note that, since $a > 0$, the mapping $(x, y) \mapsto y$ would not satisfy the condition.

13.2. Toric varieties from polytopes. A *rational polytope* P is the convex hull over a finite set of vertices $X \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n$. We can also write P as the bounded intersection of a finite collection of half spaces in $\mathbb{R}^n$, that is,

$$P = \{x \in \mathbb{R}^n \mid Ax \geq -b\}$$

for some $A \in M_{n \times n}(\mathbb{Q})$ and $b \in \mathbb{Q}^n$. A (*proper*) *face* F of P of dimension $n - k$ is the intersection with k supporting hyperplanes of the form

$$\{v \in P \mid \langle a_i, v \rangle = -b_i\},$$

where a_i is the i -th row of A . A *facet* is a face of dimension $n - 1$.

Given a rational polytope $P \in M_{\mathbb{R}} \cong \mathbb{R}^n$, define a fan Δ_P whose rays are the inward pointing normals to the facets of P . There is a cone σ_v for each vertex v of P , determined by the normals of the facets incident on v . A nice feature of this construction is the dual cone of σ_v is formed by looking at $v \in P$ and extending its edges indefinitely (and translating to the origin).

Example 13.9. Consider the square P with vertices $(0,0), (1,0), (2,1), (0,1)$ as shown in the left of Figure 32. We get $X(\Delta_P) = H_1$, the Hirzebruch surface, considered earlier, with $a = 1$.

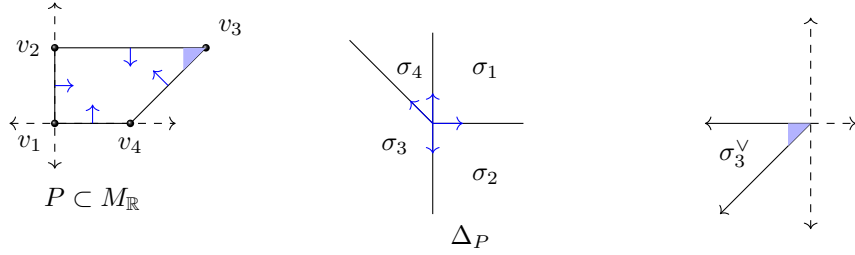


FIGURE 32. The fan Δ_P associated to a polytope P . The cones of Δ_P are in bijection with vertices of P . The dual cones are determined by the corresponding vertex figures. For instance, the blue triangle at v_3 determines the dual cone $\sigma_3^\vee$

Exercise 13.10. Let $n = 2$ and let P be the convex polytope in $M_{\mathbb{R}}$ with vertices $(-1,1), (-1,-1), (2,-1)$. What is the corresponding toric variety $X(\Delta_P)$? Describe U_σ for each $\sigma \in \Delta_P$ and the gluing maps between the two-dimensional cones of Δ_P .

13.3. Cohomology of toric varieties. We can compute the cohomology of some nice toric varieties, namely those that are *smooth* and *complete*.

A cone $\sigma \subseteq N_{\mathbb{R}}$ is *smooth* if it is generated by a subset of a $\mathbb{Z}$ -basis of N . A fan $\Delta \subseteq N_{\mathbb{R}}$ is *complete* if its support is all of $N_{\mathbb{R}}$, i.e., its fans cover $N_{\mathbb{R}}$. Suppose that Δ is a complete fan and that each cone $\sigma \in \Delta$ is smooth. In this case, it turns out that the corresponding toric variety $X(\Delta) \stackrel{\text{call}}{=} X$ is a compact orientable complex manifold.

Let $\Delta(1)$ denote the set of all 1-dimensional faces of Δ . For each $D \in \Delta(1)$, let n_D be the first lattice point along D . We define the *Chow ring* of X to be the quotient ring graded by degree:

$$A^\bullet(X) := \mathbb{Z}[D \in \Delta(1)] / (I + J),$$

where

$$\begin{aligned} I &= (\prod_{D \in S} D \mid S \subseteq \Delta(1) \text{ does not generate a cone in } \Delta), \\ J &= (\sum_{D \in \Delta(1)} \langle m, n_D \rangle D \mid m \in M), \end{aligned}$$

where $\langle m, n_D \rangle$ denotes the inner product.

The cohomology groups can be calculated through an extension of scalars:

Theorem 13.11. *Let Δ and X be as above. We have*

$$H^k X = \begin{cases} 0 & \text{if } k \text{ is odd;} \\ A^\ell(X) \otimes_{\mathbb{Z}} \mathbb{R} & \text{if } k = 2\ell. \end{cases}$$

(To form $A^\bullet(X) \otimes \mathbb{R}$, just replace $\mathbb{Z}$ by $\mathbb{R}$ in the definition. If $A^\ell(X)$ is spanned over $\mathbb{Z}$ by t monomials, then $H^{2\ell} X = \mathbb{R}^t$.)

Example 13.12. Let Δ be the fan in Example 13.6 whose associated toric variety is $X = \mathbb{P}^2$. Let $D_1 = \mathbb{R}_{\geq 0}e_1$, $D_2 = \mathbb{R}_{\geq 0}e_2$, $D_3 = \mathbb{R}_{\geq 0}(-e_1 - e_2)$. Then

$$\begin{aligned} I &= (D_1 D_2 D_3), \\ J &= (\sum_{i=1}^3 \langle e_j, n_{D_i} \rangle D_i \mid j = 1, 2) \\ &= (\langle (1, 0), (1, 0) \rangle D_1 + \langle (1, 0), (0, 1) \rangle D_2 + \langle (1, 0), (-1, -1) \rangle D_3, \\ &\quad \langle (0, 1), (1, 0) \rangle D_1 + \langle (0, 1), (0, 1) \rangle D_2 + \langle (0, 1), (-1, -1) \rangle D_3) \\ &= (D_1 - D_3, D_2 - D_3). \end{aligned}$$

We see that

$$A^\bullet(\mathbb{P}_{\mathbb{C}}^2) = \mathbb{Z}[D_1, D_2, D_3] / (D_1 D_2 D_3, D_1 - D_3, D_2 - D_3) \cong \mathbb{Z}[D_3] / (D_3^3) = \mathbb{Z} + \mathbb{Z}D_3 + \mathbb{Z}D_3^2.$$

Therefore, we have the table

$$\frac{k}{H^k(\mathbb{P}_{\mathbb{C}}^2)} \mid \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \mathbb{R} & 0 & \mathbb{R} & 0 & \mathbb{R} \end{array}.$$

Exercise 13.13. What are the cohomology groups of $\mathbb{P}^n$?

Example 13.14. Let H_a be the Hirzebruch surface as described in Example 13.8. Let $D_1 = \mathbb{R}_{\geq 0}e_1$, $D_2 = \mathbb{R}_{\geq 0}e_2$, $D_3 = \mathbb{R}_{\geq 0}(-e_1 + ae_2)$, $D_4 = \mathbb{R}_{\geq 0}(-e_2)$. Then one can check that the two ideals are

$$\begin{aligned} I &= (D_1 D_3, D_2 D_4), \\ J &= (D_1 - D_3, D_2 + aD_3 - D_4). \end{aligned}$$

Therefore, the Chow ring is

$$\begin{aligned} A^\bullet(H_a) &= \mathbb{Z}[D_1, D_2, D_3, D_4] / (D_1 D_3, D_2 D_4, D_1 - D_3, D_2 + aD_3 - D_4) \\ &\cong \mathbb{Z}[D_1, D_2] / (D_1^2, D_2(D_2 + aD_1)) \end{aligned}$$

$$= \mathbb{Z} + (\mathbb{Z}D_1 + \mathbb{Z}D_2) + \mathbb{Z}D_1D_2,$$

using the facts $D_3 = D_1$, $D_4 = D_2 + aD_3$, and $D_2^2 = -aD_1D_2$ in $A^\bullet(H_a)$. So we have the table

k	0	1	2	3	4
$H^k(H_a)$	$\mathbb{R}$	0	$\mathbb{R}^2$	0	$\mathbb{R}$

The symmetry in the above two examples is due to Poincaré duality.

13.4. Homogeneous coordinates. David Cox ([3]) introduced the homogeneous coordinates for a toric variety, generalizing those for projective space. In the case of simplicial toric varieties, we can then write the toric variety as a quotient of $\mathbb{C}^n$ modulo the action of a group, again generalizing the case of $\mathbb{P}^n = \mathbb{C}^{n+1}/(x \sim \lambda x : \lambda \in \mathbb{C}^*)$.

Let Δ be a fan in $N \cong \mathbb{Z}^n$, and let $\Delta(1)$ be the collection of all 1-dimensional faces (rays) of Δ . For each $D \in \Delta(1)$, let $n_D \in N$ be the first lattice point along D . Let $X = X(\Delta)$ be the corresponding toric variety. Define the *codimension Chow group* $A_{n-1}(X)$, (related to the Chow group $A^\bullet$, defined previously only for smooth complete toric varieties) to be the quotient

$$A_{n-1}(X) := \mathbb{Z}^{\Delta(1)} / (\sum_{D \in \Delta(1)} \langle m, n_D \rangle D \mid m \in M),$$

where $\mathbb{Z}^{\Delta(1)} = \{\sum_{D \in \Delta(1)} a_D D \mid a_D \in \mathbb{Z}\} \cong \mathbb{Z}^{|\Delta(1)|}$ is the free $\mathbb{Z}$ -module generated by $\Delta(1)$. (Note that $A_{n-1}(X) \cong A^1(X)$ if X is smooth and complete.) We have an injection of the lattice M (in which the dual cones sit) via

$$\begin{aligned} M &\rightarrow \mathbb{Z}^{\Delta(1)} \\ m &\mapsto D_m := \sum_{D \in \Delta(1)} \langle m, n_D \rangle D, \end{aligned}$$

and a short exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^{\Delta(1)} \rightarrow A_{n-1}(X) \rightarrow 0.$$

Let S be the polynomial ring

$$S := \mathbb{C}[x_D \mid D \in \Delta(1)].$$

For each $a = \sum_{D \in \Delta(1)} a_D D \in \mathbb{Z}^{\Delta(1)}$ with $a_D \geq 0$ for all D , there is a corresponding monomial

$$x^a := \prod_{D \in \Delta(1)} x_D^{a_D} \in S$$

and its *degree*, denoted $\deg(x^a)$ is defined to be its class $[a] \in A_{n-1}(X)$. Thus, the polynomial ring S is graded by $A_{n-1}(X)$.

Definition 13.15. The *homogeneous coordinate ring* of X is the polynomial ring S graded by $A_{n-1}(X)$, meaning that if we write

$$S_\alpha := \bigoplus_{\deg(x^a) = \alpha} \mathbb{C}x^a,$$

then

$$S \cong \bigoplus_{\alpha \in A_{n-1}(X)} S_\alpha,$$

and one can check that $S_\alpha \cdot S_\beta \subseteq S_{\alpha+\beta}$.

Example 13.16. The fan for $\mathbb{P}^n$ has $n+1$ rays: $D_i = \mathbb{R}e_i$ for $i = 1, \dots, n$, corresponding to the standard basis vectors, and $D_{n+1} = \mathbb{R}(-e_1 - \dots - e_n)$. To compute $A_{n-1}\mathbb{P}^n = A^1\mathbb{P}^n$, we first find

$$\begin{aligned} (\sum_{D \in \Delta(1)} \langle m, n_D \rangle D \mid m \in M) &= (\sum_{i=1}^{n+1} \langle e_j, n_{D_i} \rangle D_i \mid j = 1, \dots, n) \\ &= (D_1 - D_{n+1}, D_2 - D_{n+1}, \dots, D_n - D_{n+1}) \end{aligned}$$

Therefore,

$$\begin{aligned} A_{n-1}\mathbb{P}^n &= \mathbb{Z}^{\Delta(1)} / (D_1 - D_{n+1}, D_2 - D_{n+1}, \dots, D_n - D_{n+1}) = \mathbb{Z}D_{n+1} \cong \mathbb{Z} \\ D_i &\mapsto D_{n+1} \mapsto 1. \end{aligned}$$

Letting $x_i = x_{D_i}$ for $i = 1, \dots, n+1$, we have the homogeneous coordinate ring

$$S = \mathbb{C}[x_1, \dots, x_{n+1}].$$

The degree of a monomial $x_1^{a_1} \dots x_{n+1}^{a_{n+1}}$ is the class of $a = \sum_{i=1}^{n+1} a_i D_i$ in $A_{n-1}\mathbb{P}^n$. But in $A_{n-1}\mathbb{P}^n$, we have

$$\deg(x^a) = [a] = [\sum_{i=1}^{n+1} a_i D_i] = [(\sum_{i=1}^{n+1} a_i) D_{n+1}].$$

Identifying $A_{n-1}\mathbb{P}^n$ with $\mathbb{Z}$, as above, we get

$$\deg x^a = \sum_{i=1}^{n+1} a_i,$$

the usual degree.

Example 13.17. Let H_a denote the Hirzebruch surface as described in Example 13.8. Recall that in Example 13.14 we computed

$$\begin{aligned} A_{n-1}(H_a) &= \mathbb{Z}^{\Delta(1)} / \langle D_1 - D_3, D_2 + aD_3 - D_4 \rangle \rightarrow \mathbb{Z}\{D_1, D_2\} \\ D_1 &\mapsto D_1 \\ D_2 &\mapsto D_2 \\ D_3 &\mapsto D_1 \\ D_4 &\mapsto aD_1 + D_2 \end{aligned}$$

where $\mathbb{Z}\{D_1, D_2\} := \text{Span}_{\mathbb{Z}}\{D_1, D_2\}$. Then, finally

$$\begin{aligned} A_{n-1}(H_a) &\cong \mathbb{Z}\{D_1, D_2\} \cong \mathbb{Z}^2 \\ aD_1 + bD_2 &\mapsto (a, b). \end{aligned}$$

So the homogeneous coordinate ring for H_a is $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ and it is graded by $\mathbb{Z}^2$ with each indeterminate having degree

$$\deg(x_1) = \deg(x_3) = (1, 0), \quad \deg(x_2) = (0, 1), \quad \deg(x_4) = \deg(x_2 x_3^a) = (a, 1).$$

For example, $\deg(x_1 x_2^2 x_3^3 x_4) = (1, 0) + 2(0, 1) + 3(1, 0) + (a, 1) = (4 + a, 3)$. The monomial $x_1^2 x_2 x_3^3 x_4$ has degree $(5 + a, 2)$. The degrees of these monomials in S

differ even though they would have the same degree under the usual grading (in which each x_i had degree 1).

Quotients. Now assume that X is *simplicial*, meaning that each of its cones σ has $\dim(\sigma)$ rays. (It suffices to check the maximal-dimensional cones.) Our goal is to view X as a quotient space under some group action, generalizing the construction of projective space. Consider the following short exact sequence of $\mathbb{Z}$ -modules:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow A_{n-1}(X) \longrightarrow 0$$

$$m \longmapsto \sum_{D \in \Delta(1)} \langle m, n_D \rangle D.$$

The nonzero complex numbers, $\mathbb{C}^*$, form an abelian group under multiplication, and hence has a $\mathbb{Z}$ -module structure: for $a \in \mathbb{Z}$ and $z \in \mathbb{C}^*$, we have $a \cdot z := z^a$. It then makes sense to consider the mappings $\text{hom}_{\mathbb{Z}}(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^*)$, i.e. the $\mathbb{Z}$ -linear mappings from $\mathbb{Z}^{\Delta(1)}$ to $\mathbb{C}^*$. These are determined by the images of $D \in \Delta(1)$. For instance, if $\Delta(1) = \{D_1, \dots, D_k\}$ and $g_1, \dots, g_k$ are any elements of $\mathbb{C}^*$, there is a corresponding mapping

$$a_1 D_1 + \dots + a_k D_k \mapsto g_1^{a_1} \dots g_k^{a_k} \in \mathbb{C}^*.$$

Since the choices for the g_i are arbitrary and completely determine the mapping, we have

$$\text{hom}_{\mathbb{Z}}(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{\Delta(1)}.$$

Next, consider $\text{hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*)$. If $\Delta(1) = \{D_1, \dots, D_k\}$ and $g_1, \dots, g_k \in \mathbb{C}^*$, we can still attempt to define a mapping $A_{n-1}(X) \rightarrow \mathbb{C}^*$ by sending $D_i \mapsto g_i$, as above. However, since there are relations among the D_i in $A_{n-1}(X)$, the corresponding relations must hold among the g_i . In other words, the choices for the g_i are now constrained. For instance, if $D_3 = 2D_1 + D_2$, then since $D_3 \mapsto g_3$ and $2D_1 + D_2 \mapsto g_1^2 g_2$, we require that $g_3 = g_1^2 g_2$, i.e. $g_1^2 g_2 g_3^{-1} = 1$. In general, we define the group G by

$$G := \text{hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{C}^*) \cong \{g \in (\mathbb{C}^*)^{\Delta(1)} \mid \prod_{D \in \Delta(1)} g_D^{\langle m, n_D \rangle} = 1 \text{ for all } m \in M\}.$$

The conditions $\prod_{D \in \Delta(1)} g_D^{\langle m, n_D \rangle} = 1$, encode all the required relations. It suffices to let m range over a basis for M . So if $M = \mathbb{Z}^n$, the relations are $\prod_{D \in \Delta(1)} g_D^{\langle e_i, n_D \rangle} = 1$ for $i = 1, \dots, n$.

The inclusion

$$G \subseteq (\mathbb{C}^*)^{\Delta(1)} \subseteq \mathbb{C}^{\Delta(1)}.$$

gives a natural action of G on $\mathbb{C}^{\Delta(1)}$:

$$g \cdot x := (g_D x_D)_{D \in \Delta(1)},$$

where $g \in G, x \in \mathbb{C}^{\Delta(1)}$.

For a face $\sigma \in \Delta$, let $\sigma(1) := \Delta(1) \cap \sigma$ denote the rays in σ and define the monomial

$$x^{\hat{\sigma}} := \prod_{D \notin \sigma(1)} x_D \in S.$$

Let B be the monomial ideal

$$B := (x^{\hat{\sigma}} \mid \sigma \in \Delta) = (x^{\hat{\sigma}} \mid \sigma \text{ is a maximal cone in } \Delta) \subseteq S,$$

(note that the generators of B encode the structure of Δ) and let Z be the zero set of B ,

$$Z := \{x \in \mathbb{C}^{\Delta(1)} \mid x^{\hat{\sigma}} = 0 \text{ for all } \sigma \in \Delta\}.$$

Theorem 13.18. *Let $X = X(\Delta)$ be a simplicial toric variety. Then,*

- (1) $\mathbb{C}^{\Delta(1)} \setminus Z$ is invariant under the action by G ,
- (2) The toric variety X is the quotient of $\mathbb{C}^{\Delta(1)} \setminus Z$ by the action of G , i.e.,

$$X \cong (\mathbb{C}^{\Delta(1)} \setminus Z) / (x \sim g \cdot x \mid g \in G).$$

An element in this quotient is called the homogeneous coordinates of a point in X .

Example 13.19. Let $X = \mathbb{P}^n$. The maximal cones of the fan in this case consists of all subsets of size n from the vectors $e_1, \dots, e_n, -e_1 - \dots - e_n$. If σ is one of these cones, then it omits exactly one of these vectors. Hence,

$$B = (x_1, \dots, x_{n+1})$$

and $Z = \{0\} \subset \mathbb{C}^{n+1}$. Since $A_{n-1}\mathbb{P}^n \cong \mathbb{Z}$, we have $G = \text{hom}(A_{n-1}(\mathbb{P}^n), \mathbb{C}, \mathbb{C}^*) \cong \mathbb{C}^*$. In detail, since $A_{n-1}(\mathbb{P}^n)$ is the span of the D_i modulo the relations $D_i - D_{n+1}$ for $i = 1, \dots, n$, we have

$$\begin{aligned} G &= \{g \in \mathbb{C}^{n+1} \mid g_i g_{n+1}^{-1} = 1 \text{ for } i = 1, \dots, n\} \\ &= \{g \in \mathbb{C}^{n+1} \mid g_i = g_{n+1} \text{ for } i = 1, \dots, n\} \\ &\cong \mathbb{C}^*, \end{aligned}$$

where the isomorphism in the final step is given by $g \mapsto g_{n+1}$. Using this isomorphism, the group action on $\mathbb{C}^{\Delta(1)} = \mathbb{C}^{n+1}$ is given by

$$\lambda \cdot (x_1, \dots, x_{n+1}) = (\lambda x_1, \dots, \lambda x_{n+1})$$

where $\lambda = g_{n+1} \in \mathbb{C}^*$.

As claimed in Theorem 13.18, the set $\mathbb{C}^{\Delta(1)} \setminus Z$, i.e., $\mathbb{C}^{n+1} \setminus \{0\}$ is invariant under the action by G : if $(x_1, \dots, x_{n+1}) \neq 0$, then $\lambda(x_1, \dots, x_{n+1}) \neq 0$. Further, $\mathbb{P}^n$ is the quotient

$$\mathbb{P}^n \cong (\mathbb{C}^{n+1} \setminus \{0\}) / (x \sim \lambda x \mid \lambda \in \mathbb{C}^*).$$

Example 13.20. Now consider the Hirzebruch surface H_a . We have previously computed

$$A_{n-1}(H_a) = \mathbb{Z}^{\Delta(1)} / \langle D_1 - D_3, D_2 + aD_3 - D_4 \rangle \cong \mathbb{Z}^2.$$

From Figure 31, we see

$$B = \langle x_3 x_4, x_2 x_3, x_1 x_4, x_1 x_2 \rangle.$$

Setting all of these generating monomials equal to zero and solving, we find

$$Z = \{x \in \mathbb{C}^4 \mid x_1 = x_3 = 0 \text{ or } x_2 = x_4 = 0\}.$$

The relations for A_{n-1} are generated by $D_1 - D_3 = 0$ and $D_4 = D_2 + aD_3$. Using the first relation, the second can be rewritten as $D_4 = aD_1 + D_2$. Rewriting these multiplicatively gives

$$\begin{aligned} G &\cong \{g \in (\mathbb{C}^*)^4 \mid g_1 = g_3, g_4 = g_1^a g_2\} \cong (\mathbb{C}^*)^2 \\ g &\mapsto (g_1, g_2) \\ (g_1, g_2, g_1, g_1^a g_2) &\leftarrow (g_1, g_2). \end{aligned}$$

Therefore, the quotient description of the Hirzebruch surface is

$$H_a \cong (\mathbb{C}^4 \setminus Z) / (x \sim (\lambda x_1, \mu x_2, \lambda x_3, \lambda^a \mu x_4)),$$

where $\lambda, \mu \in \mathbb{C}^*$.

13.5. Mapping toric varieties into projective spaces. In this section, we will assume that X is a smooth, complete toric variety associated with the fan Δ . Let $\Delta(1) = \{D_1, \dots, D_\ell\}$ be the rays of Δ , and let $S = \mathbb{C}[x_1, \dots, x_\ell]$ be the homogeneous coordinate ring of X with x_i corresponding to D_i and graded by $A_{n-1}X$. Recall the short exact sequence

$$\begin{aligned} 0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow A_{n-1}(X) \longrightarrow 0 \\ m \longmapsto D_m \end{aligned}$$

where

$$(10) \quad D_m := \sum_{D \in \Delta(1)} \langle m, n_D \rangle D.$$

An element of $\mathbb{Z}^{\Delta(1)}$ is called a *divisor* of X . If all of the coefficients of a divisor are nonnegative, then the divisor is said to be *effective*. Consider an effective divisor

$$E = \sum_{D \in \Delta(1)} a_D D \in \mathbb{Z}^{\Delta(1)},$$

and associate to it the polytope

$$(11) \quad P(E) := \{m \in M_{\mathbb{R}} \mid \langle m, n_D \rangle \geq -a_D \text{ for all } D \in \Delta(1)\},$$

where we recall that n_D is the first lattice point along D . We also make the following

assumption: $X = X(\Delta_P(E))$.

Let $T = \{m_1, \dots, m_{t+1}\} \subseteq P(E) \cap M$ be a collection of lattice points in $P(E)$ containing all of its vertices. Using the homogeneous coordinates of X as described in Theorem 13.18, we get a mapping of X into projective space as follows:

$$(12) \quad \begin{aligned} \phi_T: X &\longrightarrow \mathbb{P}^t \\ x &\longmapsto (x^{D_{m_1}+E}, \dots, x^{D_{m_{t+1}}+E}), \end{aligned}$$

with D_{m_i} defined by Equation 10. The mapping ϕ_T is well-defined: Including the vertices of $P(E)$ in T assures there is no point $p \in X$ such that $\phi(T) = 0$. Further, since $[D_{m_i}] = [0] \in A_{n-1}(X)$ for all i , we have $[D_{m_i} + E] = [E]$, and the mapping is *homogeneous of degree* $[E]$. Therefore, scaling the homogeneous coordinates in the domain will scale each component of its corresponding point in the codomain.

The mapping ϕ_T will be an embedding if for each vertex v of $P(E)$, as you travel along each each emanating from v , the first lattice point you reach is an element of T .

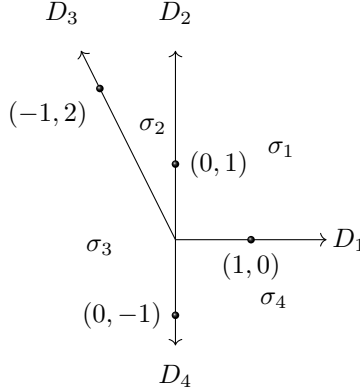
Example 13.21. Let $X = \mathbb{P}^n$ with divisors $D_i = \mathbb{R}_{\geq 0}e_i$ for $i = 1, \dots, n$ and $D_{n+1} = \mathbb{R}_{\geq 0}(-e_1 - \dots - e_n)$, as usual. Consider the effective divisor $E = dD_{n+1}$ for some positive integer d . We have

$$\begin{aligned} P(E) &= \{m \in M_{\mathbb{R}} \mid \langle m, e_i \rangle \geq 0 \text{ for } i = 1, \dots, n \text{ and } \langle m, -\sum_{i=1}^n e_i \rangle \geq -d\} \\ &= \{m \in M_{\mathbb{R}} \mid m_i \geq 0 \text{ for } i = 1, \dots, n \text{ and } \sum_{i=1}^n m_i \leq d\}, \end{aligned}$$

i.e., $P(E)$ is the simplex with vertices $0, de_1, \dots, de_n$. When $T = P(E) \cap \mathbb{Z}^n$ contains all the lattice points in $P(E)$, we get the d -uple embedding $\mathbb{P}^n \rightarrow \mathbb{P}^{\binom{n+d}{d}-1}$:

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \\ (x_0, \dots, x_n) &\longmapsto \underbrace{(x_1^d, x_1^{d-1}x_2, \dots, x_{n+1}^d)}_{\text{all monomials of degree } d}. \end{aligned}$$

Example 13.22. Let H_2 denote the Hirzebruch surface having its four rays generated by $e_1, e_2, -e_1 + 2e_2, -e_2$:



Consider the effective divisor $E = 2D_3 + 3D_4$. Taking the dot products of $m = (m_1, m_2) \in M_{\mathbb{R}} = \mathbb{R}^2$ with the first lattice points along each D_i and using Equation 11 gives the inequalities defining the corresponding polytope:

$$P(E) = \{m \in M_{\mathbb{R}} \mid m_1, m_2 \geq 0, -m_1 + 2m_2 \geq -2, -m_2 \geq -3\},$$

which is drawn in Figure 33. Consider the following set of lattice points in $P(E)$.

$$T = \{(0,0), (0,1), (0,2), (1,2), (0,3), (3,1), (8,3)\}.$$

Using Equation 10 we compute the corresponding divisors necessary for our mapping (as prescribed by Equation 12):

$$\begin{aligned} D_{(0,0)} + E &= 2D_3 + 3D_4, & D_{(0,1)} + E &= D_2 + 4D_3 + 2D_4, \\ D_{(0,2)} + E &= 2D_2 + 6D_3 + D_4, & D_{(1,2)} + E &= D_1 + 2D_2 + 5D_3 + D_4, \\ D_{(0,3)} + E &= 3D_3 + 8D_4, & D_{(3,1)} + E &= 3D_1 + D_2 + D_3 + 2D_4, \\ D_{(8,3)} + E &= 8D_1 + 3D_2. \end{aligned}$$

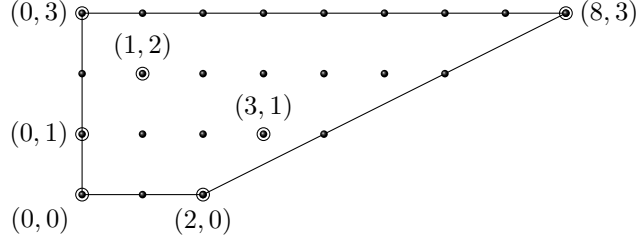


FIGURE 33. Polytope associated to the divisor $E = 2D_3 + 3D_4$ on the Hirzebruch surface H_2 . We consider the embedding of H_2 into $\mathbb{P}^6$ determined by the circled lattice points.

So the map into projective space is

$$\begin{aligned} \phi_T: H_2 &\longrightarrow \mathbb{P}^6 \\ (x_1, x_2, x_3, x_4) &\longmapsto (x_3^2 x_4^3, x_2 x_3^4 x_4^2, x_2^2 x_3^6 x_4, x_1 x_2^2 x_3^5 x_4, x_3^3 x_4^8, x_1^3 x_2 x_3 x_4^2, x_1^8 x_2^3). \end{aligned}$$

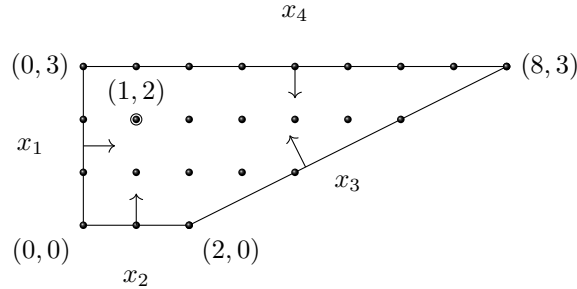
To appreciate that ϕ_T respects scaling, recall that *scaling*, in the case of H_2 is by the group $(\mathbb{C}^*)^2$ and the group action is given by

$$(\lambda, \mu) \cdot (x_1, x_2, x_3, x_4) = (\lambda x_1, \mu x_2, \lambda x_3, \lambda^2 \mu x_4).$$

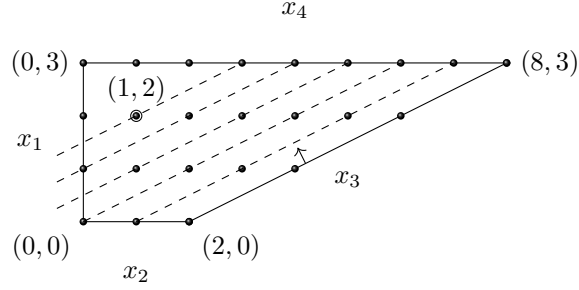
where $(\lambda, \mu) \in (\mathbb{C}^*)^2$. We then check

$$\phi_T(\lambda x_1, \mu x_2, \lambda x_3, \lambda^2 \mu x_4) = \lambda^8 \mu^3 \phi_T(x_1, x_2, x_3, x_4) = \phi_T(x_1, x_2, x_3, x_4) \in \mathbb{P}^6.$$

There is a way to read off the monomial components in ϕ_T by computing the “lattice distances” of the corresponding lattice point from each of the facets of $P(E)$. As an example, consider the lattice point $(1, 2) \in T$. Its corresponding component function in $\phi(T)$ is $x_1 x_2^2 x_3^5 x_4$. We would like to see that its exponent vector, $(1, 2, 5, 1)$ is the list of lattice distances from the facets of $P(E)$. First note that each facet of $P(E)$ corresponds to an indeterminate x_i : the inward pointing normal of a facet determines a ray D_i which determines x_i . So we label the facets accordingly:



To get to the lattice point $(1, 2)$, we need to slide the x_1 -facet over one, the x_2 -facet up two, and the x_4 -facet down one. That accounts for the first, second, and fourth components of the exponent vector $(1, 2, 5, 1)$. The following diagram illustrates that the x_3 -facet needs to be slid in its inward normal direction a total of five steps to reach $(1, 2)$:



The reader should check that, for instance, monomial $x_1^3 x_2 x_3 x_4^2$, corresponding to the lattice point $(3, 1)$, can be computed similarly.

In order to create an embedding of H_2 using the divisor $E = 2D_3 + 3D_4$, one would need to include not only the vertices in T , but also the first lattice points as you travel away from each vertex along edges incident to the vertex. In other words, T must include at least the vertices pictured in Figure 34.

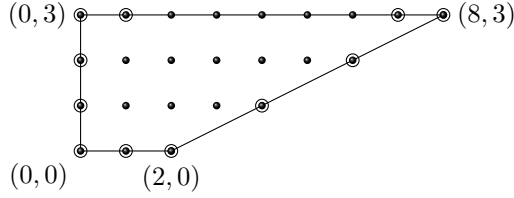


FIGURE 34. Polytope associated to the divisor $E = 2D_3 + 3D_4$ on the Hirzebruch surface H_2 . The circled lattice points are required to produce an embedding into projective space.

14. GRASSMANNIANS

Definition 14.1. If V is a vector space over $\mathbb{k}$, then *projective space* on V , denoted $\mathbb{P}(V)$, is the collection of all one-dimensional subspaces of V . A special case is $\mathbb{P}_{\mathbb{k}}^n := \mathbb{P}(\mathbb{k}^{n+1})$ (or, usually, $\mathbb{P}^n$ when $\mathbb{k}$ is clear from context). An r -plane in $\mathbb{P}(V)$ is an $(r+1)$ -dimensional subspace of V .

Proposition 14.2. Suppose L is an r -plane, M is an s -plane, and $L \cap M$ is a k -plane in $\mathbb{P}^n$. Then $k \geq r + s - n$.

Proof. Exercise. □

Duality. An $(n-1)$ -plane in $\mathbb{P}^n$ is called a *hyperplane*. It is the solution set to a linear equation

$$H_a := (a_1, \dots, a_{n+1}) \cdot (x_1, \dots, x_{n+1}) = a_1x_1 + \dots + a_{n+1}x_{n+1} = 0$$

for some $a := (a_1, \dots, a_{n+1}) \neq (0, \dots, 0) \in \mathbb{k}^{n+1}$. Thus, a is the normal vector to the hyperplane (in the case $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$). Further, a is determined by the hyperplane up to scaling by a nonzero element of $\mathbb{k}$. Define $(\mathbb{P}^n)^*$ to be the *dual projective space* whose points are the hyperplanes in $\mathbb{P}^n$. Then we get a well-defined bijection between $\mathbb{P}^n$ and its dual:

$$\begin{aligned} \mathbb{P}^n &\rightarrow (\mathbb{P}^n)^* \\ a &\mapsto H_a. \end{aligned}$$

Definition 14.3. Let V be a vector space over $\mathbb{k}$. The collection of $(r+1)$ -dimensional subspaces of V is called a *Grassmannian* and denoted by $G(r+1, V)$. It is also called the Grassmannian of r -planes in $\mathbb{P}(V)$ and then denoted by $\mathbb{G}_r\mathbb{P}(V)$. If $V = \mathbb{k}^{n+1}$, the Grassmannian is denoted by either $G(r+1, n+1)$ or $\mathbb{G}_r\mathbb{P}^n$. (We say $\mathbb{G}_r\mathbb{P}^n$ is a *moduli space* for the set of r -planes in $\mathbb{P}^n$ since it parametrizes this set.)

Example 14.4. Some special cases of Grassmannians:

$$\mathbb{G}(1, n+1) = \mathbb{G}_0\mathbb{P}^n = \mathbb{P}^n, \quad \mathbb{G}(2, 3) = \mathbb{G}_1\mathbb{P}^2 = (\mathbb{P}^2)^*, \quad \mathbb{G}_{n-1}\mathbb{P}^n = (\mathbb{P}^n)^*.$$

The Grassmannian $G(2, 4) = \mathbb{G}_1\mathbb{P}^3$ is the set of lines in three-space.

14.1. Manifold structure. We now fix our vector space $\mathbb{k}$ to be $\mathbb{C}$ (although a lot of what we do below will carry over $\mathbb{k} = \mathbb{R}$ or to arbitrary fields). Given $L \in G(r, n) = \mathbb{G}_{r-1}\mathbb{P}^{n-1}$, we can write

$$L = \text{Span}\{a_1, \dots, a_r\}$$

for some vectors $a_1, \dots, a_r \in \mathbb{C}^n$. Let A be the matrix whose rows are $a_1, \dots, a_r$. Then $L = \text{Span}\{b_1, \dots, b_r\}$ for some other vectors b_i if and only if there is an invertible $r \times r$ matrix M such that $B = MA$, where B is the matrix whose rows are the b_i . (Multiplying A by M on the left performs invertible row operations and, thus, does not change the rowspan.) Therefore,

$$G(r, n) = \{r \times n \text{ rank } r \text{ matrices}\} / (A \sim MA : M \text{ } r \times r, \text{ invertible}).$$

Identifying the set of $r \times n$ matrices with $\mathbb{C}^{r \times n} = \mathbb{C}^{rn}$ induces a topology on the set of $r \times n$ matrices and the quotient topology on $G(r, n)$. (So a subset of $G(r, n)$ is open if and only if the set of all matrices representing points in that set forms an open subset of $\mathbb{C}^{r \times n}$.) The case $r = 1$ recovers the usual construction of projective space— A will be $1 \times n$ and $M = [\lambda]$ for some nonzero $\lambda \in \mathbb{C}$. In general, we consider an $r \times n$ matrix of rank r to be the *homogeneous coordinates* for a point in $G(r, n)$.

We now seek an open covering of $G(r, n)$ and chart mappings generalizing those for projective space. We motivate the idea with an example:

Example 14.5. Consider

$$L = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{pmatrix} \in G(2, 4) = \mathbb{G}_1\mathbb{P}^3,$$

a line in $\mathbb{P}^3$. With respect to the chart (U_4, ϕ_4) on $\mathbb{P}^3$, one may check that the line is parametrized by

$$t \mapsto (1, 0, 3) + t((2, 4, 3) - (1, 0, 3)),$$

i.e., the line containing the points $(1, 0, 3)$ and $(2, 4, 3)$. To find what we will soon define to be coordinates for L with respect to a particular chart for $G(2, 4)$, choose any two linearly independent columns and perform row operations to reduce that pair of columns to the identity matrix. Choosing the first and fourth columns gives:

$$\begin{array}{ccccccc} \star & & \star & & & & \\ \left(\begin{array}{cccc} 1 & 0 & 3 & 1 \\ 2 & 4 & 3 & 1 \end{array} \right) & \rightarrow & \left(\begin{array}{cccc} 1 & 0 & 3 & 1 \\ 0 & 4 & -3 & -1 \end{array} \right) & \rightarrow & \left(\begin{array}{cccc} 1 & 4 & 0 & 0 \\ 0 & 4 & -3 & -1 \end{array} \right) & \rightarrow & \left(\begin{array}{cccc} 1 & 4 & 0 & 0 \\ 0 & -4 & 3 & 1 \end{array} \right). \\ L & & & & & & L' \end{array}$$

Performing the same row operations to the identity matrix I_2 yields the matrix

$$M = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix},$$

and $ML = L'$. Since the operations are invertible, all matrices row equivalent to L' are equivalent to each other, i.e., represent the same point in $G(2, 4)$. Knowing that the columns 1 and 4 have been fixed, the point with homogeneous coordinates L can then be uniquely represented by the entries in columns 2 and 3. If we then agree to read the entries of a matrix from left-to-right and top-to-bottom, we assign the unique point $(4, 0, -4, 3) \in \mathbb{C}^4$ to L . These are the coordinates of L with respect to columns 1 and 4. Every 2×4 matrix whose first and fourth columns are linearly independent will have similar coordinates. Also, every point $x \in \mathbb{C}^4$ has a corresponding point in $\mathbb{G}_1\mathbb{P}^3$ represented by a matrix whose first and fourth columns form I_2 and whose second and third columns are given by the coordinates of x . The result is a chart, which we will call $(U_{1,4}, \phi_{1,4})$.

We generalize the above example to create an atlas for $G(r, n)$. Let $j \in \mathbb{Z}^r$ with $1 \leq j_1 < \dots < j_r \leq n$. Given an $r \times n$ matrix L , let L_j be the square $r \times r$

submatrix of L_j formed by the columns with indices $j_1, \dots, j_r$. Then define

$$U_j := \{L \in G(r, n) \mid \text{rk } L_j = r\}.$$

Note that the rank of $L \in G(r, n)$ is independent of the choice of representative matrix. Define the corresponding chart mapping by

$$(13) \quad \begin{aligned} \phi_j: U_j &\rightarrow \mathbb{C}^{r(n-r)} \\ L &\mapsto \text{flatten}_j(L_j^{-1}L), \end{aligned}$$

where flatten_j creates a point in $\mathbb{C}^{r(n-r)}$ from an $r \times r$ matrix by reading off the entries in the columns not indexed by j from left-to-right and top-to-bottom. Multiplying L by L_j^{-1} on the left performs row operations on L so that the resulting matrix has I_r as the submatrix indexed by j . To get the coordinates with respect to ϕ_j , we then just read off the other entries in the matrix.

Proposition 14.6. *Using the notation defined above, $\{(U_j, \phi_j)\}_j$ is an atlas for $G(r, n)$:*

- (1) *each U_j is open in $G(r, n)$;*
- (2) *the U_j cover $G(r, n)$; and*
- (3) *$\phi_j: U_j \rightarrow \mathbb{C}^{r(n-r)}$ is a homeomorphism.*

Proof. Let $\mathbb{C}^{a \times b}$ denote that set of $a \times b$ matrices. Fix the isomorphism $\mathbb{C}^{a \times b} \cong \mathbb{C}^{ab}$ by reading the entries of a matrix from left-to-right and top-to-bottom. The topology on $\mathbb{C}^{a \times b}$ is determined by insisting the isomorphism is a homeomorphism. For each j , let π_j be the projection mapping that sends an $r \times n$ matrix L to the $r \times r$ submatrix L_j . Then we have a sequence of continuous mappings

$$\begin{aligned} \mathbb{C}^{r \times n} &\xrightarrow{\pi_j} \mathbb{C}^{r \times r} \xrightarrow{\det} \mathbb{C} \\ L &\mapsto L_j \mapsto \det(L_j). \end{aligned}$$

The composition is continuous, so the inverse image of $0 \in \mathbb{C}$ is closed in $\mathbb{C}^{r \times n}$. Call the complement of this set $\tilde{U}_j$. Then $\tilde{U}_j$ is open, and the quotient of $\tilde{U}_j$ modulo our equivalence $A \sim MA$ defining the Grassmannian is U_j . Since we have given the Grassmannian the quotient topology, we conclude that U_j is open.

A point in $G(r, n)$ is represented by an $r \times n$ matrix L of rank r . Row reducing L must then produce I_r as a square submatrix. Let j be the indices of the columns of that submatrix in L . Then $L \in U_j$.

We now consider a chart mapping $\phi_j: U_j \rightarrow \mathbb{C}^{r(n-r)}$. First note that $L \mapsto L_j^{-1}L$ is a continuous function of L : use the formula for the inverse using cofactors to see that the inverse is a rational function of the entries of the matrix. Then, since matrix multiplication and projections are continuous, it follows that ϕ_j is continuous. The inverse is the continuous mapping that sends a point in $p \in \mathbb{C}^{r(n-r)}$ to the matrix L such that $L_j = I_r$ and whose other entries are obtained from p . $\square$

Definition 14.7. The *standard atlas* for $G(r, n)$ is $\{(U_j, \phi_j)\}_j$ where $j = (j_1, \dots, j_r)$ with $1 \leq j_1 < j_2 < \dots < j_r \leq n$, and ϕ_j is defined in Equation 13.

Example 14.8. If $L \in U_{1,2,4} \subset G(3, 7)$, then it has a representative of the form

$$\begin{pmatrix} 1 & 0 & * & 0 & * & * & * \\ 0 & 1 & * & 0 & * & * & * \\ 0 & 0 & * & 1 & * & * & * \end{pmatrix}.$$

The $3(7-3) = 12$ entries denoted by asterisks give the coordinates of the point via the chart $(U_{1,2,4}, \phi_{1,2,4})$.

Exercise 14.9. The set of lines in three-space is $\mathbb{G}_1\mathbb{P}^3$, and $\dim \mathbb{G}_1\mathbb{P}^3 = 4$. Now, a line is determined by a point and a direction. So that sounds like six parameters should be necessary. Give an intuitive explanation for the fact that the set of lines in three-space should be four-dimensional.

14.2. The Plücker embedding.

Definition 14.10. Fix an ordering of the $\binom{n}{r}$ choices of column indices: $j: 1 \leq j_1 < \dots < j_r \leq n$. The *Plücker embedding* is defined by

$$\begin{aligned} \Lambda: G(r, n) &\rightarrow \mathbb{P}^{\binom{n}{r}-1} \\ L &\mapsto (\det(L_j))_j, \end{aligned}$$

where L_j is the submatrix of L with columns indexed by j . The components of $\Lambda(L)$ are the *Plücker coordinates* of L . The j -th *Plücker coordinate* of L is $\det(L_j)$.

Example 14.11. The Plücker embedding of $G(2, 4)$ is

$$\begin{aligned} \Lambda: G(2, 4) &\rightarrow \mathbb{P}^5 \\ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} &\mapsto (a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_1b_4 - a_4b_1, a_2b_3 - a_3b_2, a_2b_4 - a_4b_2, a_3b_4 - a_4b_3). \end{aligned}$$

Proposition 14.12. *The Plücker coordinates of $L \in G(r, n)$ are well-defined.*

Proof. Since (any representative of) $L \in G(r, n)$ has rank r , it follows that $\det(L_j) \neq 0$ for some j . Next, suppose that A and B are $r \times n$ matrices both representing L . Then there exists an $r \times r$ matrix M such that $B = MA$. Therefore, for each choice j of r columns, we have $B_j = MA_j$. Hence, $\det B_j = \det M \det A_j$ for all j . So $(B_j)_j = \lambda(A_j)_j$ where $\lambda = \det M \neq 0$, and $\Lambda(A) = \Lambda(B) \in \mathbb{P}^{\binom{n}{r}-1}$. $\square$

Coordinate-free description. Let $G(r, V)$ denote the set of r -dimensional subspaces of a vector space V . Define the Plücker embedding of $G(r, V)$ by

$$\begin{aligned} G(r, V) &\rightarrow \mathbb{P}(\Lambda^r V) \\ W &\mapsto \Lambda^r W \end{aligned}$$

Picking a basis for V recovers the Plücker embedding defined previously.

The Plücker relations. We now seek the equations defining the image of the Plücker embedding. Denote the coordinates of a point in $\mathbb{P}^{\binom{n}{r}}$ by

$$x(j) = x(j_1, \dots, j_r)$$

for each $j: 1 \leq j_1 < \dots < j_r \leq n$. For notational convenience, we adopt the following conventions to allow permutations of j and repetitions of the indices:

$$\begin{aligned} x(\sigma(j)) &:= \text{sign}(\sigma)x(j) && \text{for each } \sigma \in \mathfrak{S}_r, \text{ and} \\ x(j) &:= 0 && \text{if } j_s = j_t \text{ for some } s \neq t. \end{aligned}$$

Definition 14.13 (Plücker relations). For each $I: 1 \leq i_1 < \dots < i_{r-1} \leq n$ and $J: 1 \leq j_1 < \dots < j_{r+1} \leq n$, the I, J -th Plücker relation is

$$P_{I,J} := \sum_{k=1}^{r+1} (-1)^k x(i_1, \dots, i_{r-1}, j_k) x(j_1, \dots, \widehat{j_k}, \dots, j_{r+1}).$$

Example 14.14. For $G(2, 4)$ with $I = \{1\}$ and $J = \{2, 3, 4\}$,

$$P_{I,J} = -x(1, 2)x(3, 4) + x(1, 3)x(2, 4) - x(1, 4)x(2, 3).$$

In this case, all other choices for I and J yield this same relation up to sign. Therefore, the image of this Grassmannian under its Plücker embedding is

$$\Lambda(G(2, 4)) = \{x \in \mathbb{P}^5 \mid P_{\{1\}, \{2, 3, 4\}} = 0\},$$

a quadric hypersurface in $\mathbb{P}^5$. The reader may want to check that the coordinates displayed in Example 14.11 satisfy this relation.

Theorem 14.15. *The Plücker embedding $\Lambda: G(r, n) \rightarrow \mathbb{P}^{\binom{n}{r}}$ is injective with image equal to the zero-set of $\{P_{I,J}\}_{I,J}$.*

Proof. We will just show that the image satisfies $P_{I,J} = 0$ for each I, J . For injectivity, one just back-solves the relations. See [9] for details. Let $L = (a_{ij})$ be an $r \times n$ matrix representing a point in $G(r, n)$. In the following calculation, we expand the first (red) determinant along its last row; the expression in the penultimate step evaluates to zero since the final (blue) matrix has a repeated row:

$$\begin{aligned} P_{I,J}(\Lambda(L)) &= \sum_{k=1}^{r+1} (-1)^k \det L_{i_1, \dots, i_{r-1}, j_k} \det L_{j_1, \dots, \widehat{j_k}, \dots, j_{r+1}} \\ &= \sum_{k=1}^{r+1} (-1)^k \underbrace{\begin{vmatrix} \dots & a_{1j_k} \\ \vdots & \vdots \\ a_{r,j_k} \end{vmatrix}}_{\text{expand}} \begin{vmatrix} \widehat{a_{1j_k}} & \dots & \widehat{a_{rj_k}} \end{vmatrix} \\ &= \pm \sum_{k=1}^{r+1} (-1)^k \sum_{\ell=1}^r (-1)^\ell a_{\ell j_k} \begin{vmatrix} \vdots & \vdots & \vdots \\ \widehat{a_{\ell i_1}} & \dots & \widehat{a_{\ell i_{r-1}}} \\ \vdots & \vdots & \vdots \end{vmatrix} \begin{vmatrix} \widehat{a_{1j_k}} & \dots & \widehat{a_{rj_k}} \end{vmatrix} \\ &= \pm \sum_{\ell=1}^r (-1)^\ell \begin{vmatrix} \vdots & \vdots & \vdots \\ \widehat{a_{\ell i_1}} & \dots & \widehat{a_{\ell i_{r-1}}} \\ \vdots & \vdots & \vdots \end{vmatrix} \left(\sum_{k=1}^{r+1} (-1)^k a_{\ell j_k} \begin{vmatrix} \widehat{a_{1j_k}} & \dots & \widehat{a_{rj_k}} \end{vmatrix} \right) \end{aligned}$$

$$\begin{aligned}
&= \pm \sum_{\ell=1}^r (-1)^\ell \left| \begin{array}{ccc} \vdots & & \vdots \\ \widehat{a_{\ell i_1}} & \cdots & \widehat{a_{\ell i_{r-1}}} \\ \vdots & & \vdots \end{array} \right| \left(\begin{array}{ccccc} a_{\ell j_1} & \cdots & a_{\ell j_k} & \cdots & a_{\ell j_{r+1}} \\ a_{1 j_1} & \cdots & a_{1 j_k} & \cdots & a_{1 j_{r+1}} \\ & & \vdots & & \\ a_{r j_1} & \cdots & a_{r j_k} & \cdots & a_{r j_{r+1}} \end{array} \right) \\
&= 0.
\end{aligned}$$

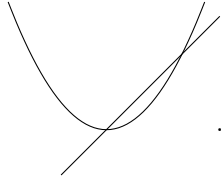
□

14.3. Schubert varieties. Our goal is to describe the cohomology ring of a Grassmannian. In this section we describe sets whose classes will give a basis for that ring. There are many closely-related cohomology theories. We have carefully considered de Rham cohomology. Appendix D is an introduction to abstract simplicial homology, which can be dualized (by applying the function $\text{hom}(\cdot, R)$) to define cohomology groups). It is fairly straightforward to relate abstract simplicial cohomology to a cohomology theory for manifolds by, roughly, triangulating a manifold. In section 13, we gave an operational definition of the Chow ring of a smooth complete toric variety, and that will be the approach we take with Grassmannians. We will give a very rough idea of the construction of the Chow ring, in general, and give a recipe for computing it for the Grassmannian. We will then describe in detail the meaning of the Chow classes for the Grassmannian and use calculations in the Chow ring to solve interesting enumerative problems.

14.3.1. The Chow ring. An *algebraic set* is the set of solutions to a system of polynomial equations. An algebraic set is *irreducible* if it cannot be written as a proper union of algebraic sets. An irreducible algebraic set is called a *variety*.

Example 14.16. The algebraic set $X = \{(x, y) \in \mathbb{R}^2 \mid x(y - x^2) = 0\}$ is reducible. It can be written as the union of the y -axis and a parabola, both of which are varieties:

$$X = \{(x, y) \in \mathbb{R}^2 \mid x = 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid y - x^2 = 0\},$$



If our defining polynomials come from the polynomial ring $\mathbb{k}[x_1, \dots, x_n]$, then the corresponding algebraic set is a subset of $\mathbb{k}^n$. If X is defined by the polynomials $f_1, \dots, f_k$, we write $X = Z(f_1, \dots, f_k)$. If the defining polynomials are homogeneous, then we may regard X as a subset of projective space.

From now on, we will take $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$. When working over $\mathbb{C}$, there is the option of replacing $\mathbb{C}$ by $\mathbb{R}^2$ to define an algebraic set over $\mathbb{R}$.

Example 14.17. Consider the algebraic set $X = \{(w, z) \in \mathbb{C}^2 \mid w = z^2\}$. This is a variety in $\mathbb{C}^2$, but since $\mathbb{C} = \mathbb{R}^2$, we may also consider it as a variety in $\mathbb{R}^4$. Letting $w = a + bi$ and $z = c + di$ and equating real and imaginary parts in $w = z^2$, then $X \subset \mathbb{R}^4$ is defined by the system of equations

$$a = c^2 - d^2 \quad \text{and} \quad b = 2cd.$$

Thus, considering X as a subset of $\mathbb{R}^4$, it is the intersection of two quadric hypersurfaces¹¹ giving a (two-dimensional) surface in $\mathbb{R}^4$.

There are many ways to define the dimension of a variety $X \in \mathbb{A}^n$. We will state one possibility that does not require a long digression. Say X is defined by the system of equations $f_1 = \cdots = f_k = 0$,

$$\dim X = n - \max_{p \in X} \operatorname{rk} \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j}.$$

A point p at which the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j} \right)_{i,j}$ reaches its maximum rank is a *smooth* point, and a non-smooth point is a *singular point*. The set of smooth points of X will form a manifold.

Example 14.18. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = xy\}$, a cone, with a single defining equation $f = z^2 - xy$. The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} -y & -x & 2z \end{pmatrix}.$$

It has maximum rank of 1, which occurs at all points of X except $(0, 0, 0)$. The dimension of X is $3 - 1 = 2$, and $(0, 0, 0)$ is the only singular point.

Let X be a variety. An r -cycle of X is a finite formal sum $\sum_i n_i V_i$ where $n_i \in \mathbb{Z}$ and V_i is an r -dimensional subvariety of X . Let $Z_r(X)$ denote the set of all r -cycles, an abelian group under addition. A subvariety $V \subseteq X$ has *codimension* r if $\dim V = \dim X - r$. So Y is a hypersurface in X if it has codimension one. There is a notion of *rational equivalence* of subvarieties, which we will not define, but which is an algebraic version of homotopy equivalence. If subvarieties V and W are rationally equivalent, we will write $V \sim W$. Rational equivalence preserves many essential properties. For instance, if $V \sim W$, then $\dim V = \dim W$.

Definition 14.19. Suppose that X is a smooth variety of dimension n . Let $A^r = Z_{n-r}(X)/\sim$. The *Chow ring* of X is $A^\bullet(X) = \bigoplus_{r \geq 0} A^r(X)$.

To define the ring structure on $A^\bullet(X)$, let $[V] \in A^r(X)$ and $[W] \in A^s(X)$. Then let $[V] \cdot [W] = [V \cap W] \in A^{r+s}(X)$ where the intersection is done after deforming V and W (replacing V and W with rationally equivalent subvarieties) so that V and W meet transversally everywhere. To meet *transversally* means that the tangent spaces of V and W at each point p of their intersection together span the tangent space of X .

¹¹By *hypersurface*, we mean codimension one in the ambient space.

Define the degree of a variety X in $\mathbb{A}^n$ or $\mathbb{P}^n_{\mathbb{C}}$ is the number of points in the intersection (over the complex numbers) of X with a generic linear space L of complementary dimension, i.e., such that $\dim L = n - \dim X$.

Example 14.20. Let $X = \mathbb{P}^n$. It turns out that

$$\begin{aligned} A^\bullet(X) &\xrightarrow{\cong} \mathbb{Z}[t]/(t^{n+1}) = \mathbb{Z} + \mathbb{Z}t + \cdots + \mathbb{Z}t^n \\ V &\mapsto (\deg V) t^{\operatorname{codim} V}. \end{aligned}$$

For example, let $n = 2$, and take $p, q \in \mathbb{P}^2$, $X = Z(yz - x^2)$, and $Y = Z(zy^2 - x^3 - zx^2)$. Then we have the following examples of calculations in the Chow ring:

- $2[p] + 3[q] - [X] + 4[Y] + 7[\mathbb{P}^2] \mapsto 2t^2 + 3t^2 - 2t + 4(3t) + 7 = 7 + 10t + 5t^2$.
- $(3[p] + [X] + [\mathbb{P}^2])^2 \mapsto (3t^2 + 2t + 1)^2 = 1 + 4t + 10t^2 + 12t^3 + 9t^4$.
- $[X][Y] \mapsto (2t)(3t) = 6t^2$.

The class of a point in $\mathbb{P}^2$ is given by t^2 . Hence, the last calculation shows that X and Y meet in 6 points after continuous deformation.

14.3.2. *Schubert varieties.* For details regarding the following, see [9]. Throughout, when necessary, we assume that $\mathbb{G}_r\mathbb{P}^n$ is embedded in $\mathbb{P}^{\binom{n}{r}}$ via the Plücker embedding.

Definition 14.21. A sequence $A_0 \subsetneq \cdots \subsetneq A_r$ of linear subspaces of $\mathbb{P}^n$ is called a *flag*. The *Schubert variety* corresponding to a flag is

$$\mathfrak{S}(A_0, \dots, A_r) = \{L \in \mathbb{G}_r\mathbb{P}^n \mid \dim(L \cap A_i) \geq i \text{ for all } i\}.$$

Proposition 14.22. *We have*

$$\mathfrak{S}(A_0, \dots, A_r) = \mathbb{G}_r\mathbb{P}^n \cap M$$

for some linear subspace M of $\mathbb{P}^{\binom{n+1}{r+1}-1}$. The space M is a hyperplane if and only if $\dim A_0 = n - r - 1$ and $\dim A_i = n - r + i$ for $i = 1, \dots, r$. (So $(A_r, A_{r-1}, \dots, A_1) = (n, n-1, \dots, n-r+1)$ and $\dim A_0 = n - r - 1$ is one less than expected given the previous sequence.)

Proposition 14.23. *If $A_0 \subsetneq \cdots \subsetneq A_r$ and $B_0 \subsetneq \cdots \subsetneq B_r$ are flags in $\mathbb{P}^n$ with $\dim A_i = \dim B_i$ for all i , then there is an invertible linear transformation of $\mathbb{P}^{\binom{n+1}{r+1}-1}$ that sends $\mathfrak{S}(A_0, \dots, A_r)$ to $\mathfrak{S}(B_0, \dots, B_r)$, and*

$$[\mathfrak{S}(A_0, \dots, A_r)] = [\mathfrak{S}(B_0, \dots, B_r)] \in A^\bullet \mathbb{G}_r\mathbb{P}^n.$$

Definition 14.24. If $\dim A_i = a_i$ for all i , then we write $\mathfrak{S}(a_0, \dots, a_r)$ or just $(a_0, \dots, a_r)$ for the class $[\mathfrak{S}(A_0, \dots, A_r)] \in A^\bullet \mathbb{G}_r\mathbb{P}^n$. The $(a_0, \dots, a_r)$ are called *Schubert cycles* or *Schubert classes*.

Theorem 14.25. *The Chow ring $A^\bullet \mathbb{G}_r\mathbb{P}^n$ is a free abelian group on the Schubert cycles $(a_0, \dots, a_r)$ (i.e., it is generated by the Schubert cycles, and there are no nontrivial integer combinations of the Schubert cycles that are 0). The codimension of $(a_0, \dots, a_r)$ is $k := (r+1)(n-r) - \sum_i (a_i - i)$, i.e., $(a_0, \dots, a_r) \in A^k \mathbb{G}_r\mathbb{P}^n$.*

The above result might be considered the solution to Hilbert's 15th problem ([7]):

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.

Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.

Example 14.26. Consider the Grassmannian of planes in five-space, $\mathbb{G}_2\mathbb{P}^5$. Here, we consider the meaning of some Schubert varieties.

I. Let $(a_0, a_1, a_2) = (1, 3, 4)$. Then

$$\mathfrak{S}(A_0, A_1, A_2) = \{L \in \mathbb{G}_2\mathbb{P}^5 \mid \dim(L \cap A_i) \geq i \text{ for all } i = 0, 1, 2\}.$$

The meaning is given by

$$\dim L \cap A_0 \geq 0 \Rightarrow L \text{ meets the line } A_0 \text{ in at least a point}$$

$$\dim L \cap A_1 \geq 1 \Rightarrow L \text{ meets the solid } A_1 \text{ in at least a line}$$

$$\dim L \cap A_2 \geq 2 \Rightarrow L \text{ meets the 4-plane } A_2 \text{ in at least a plane.}$$

Since the dimension of L is 2, the last condition means that L lies in the 4-plane A_2 .

II. Let $(a_0, a_1, a_2) = (3, 4, 5)$. Then $\mathfrak{S}(A_0, A_1, A_2)$ is those planes L in $\mathbb{P}^5$ such that

$$\dim L \cap A_0 \geq 0, \quad \dim L \cap A_1 \geq 1 \quad \text{and} \quad \dim L \cap A_2 \geq 2.$$

However, by Proposition 14.2, these conditions are satisfied by all planes L . So there is not condition at all, and $\mathfrak{S}(A_0, A_1, A_2) = \mathbb{G}_2\mathbb{P}^5$.

III. Let $(a_0, a_1, a_2) = (1, 4, 5)$. Then $\mathfrak{S}(A_0, A_1, A_2)$ represents the condition that a 2-plane meets a given line in at least a point, i.e., the plane intersects a given line. The conditions $\dim L \cap A_1 \geq 1$ and $\dim L \cap A_2 \geq 2$ are non-conditions by Proposition 14.2.

When is it the case that $\dim L \cap A_i \geq i$ imposes no condition, i.e., when is it the case that every r -plane L satisfies this restriction? By Proposition 14.2,

$$\dim L \cap A_i \geq \dim L + \dim A_i - n = r + a_i - n$$

for all L and A_i . Therefore, $\dim L \cap A_i \geq i$ for all $L \in \mathbb{G}_r\mathbb{P}^n$ if $r + a_i - n \geq i$, i.e., if

$$a_i \geq n - r + i.$$

For instance any of $a_r \geq n$, or $a_{r-1} \geq n-1$, or $a_{r-2} \geq n-2$ all impose no condition. Since $A_0 \subsetneq \cdots \subsetneq A_r \subseteq \mathbb{P}^n$, we have $0 \leq a_0 < \cdots < a_r \leq n$. So if $a_i = n - r + i$ for some i , then $a_j = n - r + j$ for all $j \geq i$.

Example 14.27. In $\mathbb{G}_3\mathbb{P}^6$

$$(3, 4, 5) = [\mathbb{G}_3\mathbb{P}^6].$$

In $(2, 4, 5)$, the entries $a_1 = 4$ and $a_2 = 5$ impose no conditions. So the only condition of consequence is that $\dim L \cap A_0 \geq 0$. This is the condition that the solid (i.e., the 3-plane) L intersects the plane A_0 . (In 6-space, there is just room for a solid and a plane to not meet.) Similarly, in $(1, 3, 5)$, the entry $a_2 = 5$ imposes no condition. So $(1, 3, 5)$ can be thought of as the class in the Chow ring represented by all solids L that intersects a given line ($\dim L \cap A_0 \geq 0$) and meet a given solid in at least a line ($\dim L \cap A_1 \geq 1$).

Example 14.28. A basis for the Chow ring $A^\bullet\mathbb{G}_1\mathbb{P}^3$ consist of the Schubert classes (a_0, a_1) with $0 \leq a_0 < a_1 \leq 3$. Here is a table of these classes and their interpretation (the alternate notation for the class using curly braces is for use in the next section):

codimension	class	condition
0	$(2, 3) = \{0, 0\}$	no condition
1	$(1, 3) = \{1, 0\}$	meet a given line
2	$(0, 3) = \{2, 0\}$	pass through a given point
2	$(1, 2) = \{1, 1\}$	lie in a given plane
3	$(0, 2) = \{2, 1\}$	pass through a given point and lie in a given plane
4	$(0, 1) = \{2, 2\}$	to be a certain line.

For instance, the last class, $(0, 1)$, imposes the conditions $\dim L \cap A_0 \geq 0$ and $\dim L \cap A_1 \geq 1$. So L must contain the point A_0 and must meet the line A_1 in a space of dimension 1, i.e., L must equal A_1 , at which point L contains the point A_0 automatically.

14.3.3. The Schubert calculus. In this section, we describe the ring structure for $A^\bullet\mathbb{G}_r\mathbb{P}^n$. Given a Schubert class $(a_0, \dots, a_r)$, define the integers

$$\lambda_i = n - r - (a_i - i)$$

for $i = 0, \dots, r$. Since $0 \leq a_0 < \dots < a_r \leq n$, it follows that each $a_i - i \geq 0$ and

$$n - r \geq \lambda_0 \geq \dots \geq \lambda_r \geq 0.$$

Further,

$$|\lambda| := \sum_{i=0}^r \lambda_i = (r+1)(n-r) - \sum_{i=0}^r (a_i - i) = \text{codim}(a_0, \dots, a_r).$$

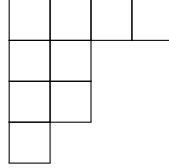
Thus,

$$\{\lambda_0, \dots, \lambda_r\} \in A^{|\lambda|}\mathbb{G}_r\mathbb{P}^n.$$

Notation: We denote the Schubert class $(a_0, \dots, a_r)$ by $\{\lambda_0, \dots, \lambda_r\}$.

For each λ there is an associated *Young diagram* consisting of rows of left-justified unit boxes such that row i has λ_i boxes. The following example should make the definition clear.

Example 14.29. The Schubert class $(1, 4, 5, 7) = \{4, 2, 2, 1\} \in \mathbb{G}_3\mathbb{P}^8$ has Young diagram



The codimension is $|\{4, 2, 2, 1\}| = 9$ and the dimension of $\mathbb{G}_3\mathbb{P}^8$ is $(3+1)(8-3) = 20$. Thus, a representative Schubert variety $\mathfrak{S}(A_0, \dots, A_r)$ has dimension $20 - 9 = 11$.

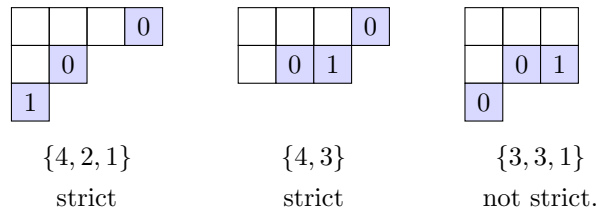
The product of two Schubert classes $\{\lambda\}$ and $\{\mu\}$ will be a unique integer linear combination of Schubert classes. So we can write

$$\{\lambda\} \cdot \{\mu\} = \sum_{\{\nu\} \in A^\bullet \mathbb{G}_r \mathbb{P}^n} c_{\lambda\mu}^\nu \{\nu\}.$$

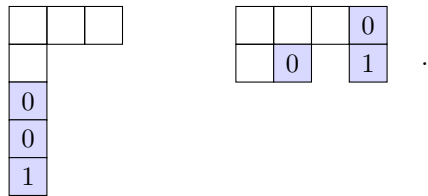
The integers $c_{\lambda\mu}^\nu$ are called *Littlewood-Richardson coefficients*. It was shown in 2006 that the general problem of computing Littlewood-Richardson numbers is $\#$ -P complete. Nevertheless, there are several ways of calculating them, of which we now describe one.

Definition 14.30. Let $\{\lambda\}, \{\mu\} \in A^\bullet \mathbb{G}_r \mathbb{P}^n$. Form the μ -*expansion* of λ in steps as follows. Let Y be the Young diagram for λ . Use μ to build a new Young diagram in steps from Y as follows: At the i -th step, add μ_i boxes to existing rows or columns with the restriction that no two of these boxes can be added to the same column. Next, write the number i in each of the added boxes. The resulting shape must be a Young diagram. Then proceed to step $i + 1$ (stopping with the last part of μ is reached). The expansion is *strict* if reading off the numbers right-to-left and top-to-bottom, each number i occurs at least as many times as the number $i + 1$.

Example 14.31. Some examples of $\{2, 1\}$ expansions of $\{3, 1\}$:



The following expansions are not allowed—the first because there are two boxes with 0s in the same column, and the second because it is not a Young diagram:



Theorem 14.32. *The Littlewood-Richardson number $c_{\lambda\mu}^\nu$ is the number of strict μ -expansions of λ resulting in ν .*

From the theorem, it is easy to see that $c_{\lambda\mu}^\nu = 0$ unless $\lambda, \mu \subseteq \nu$, i.e., unless $\lambda_i \leq \nu_i$ and $\mu_i \leq \nu_i$ for all i . It is a little harder to see that $c_{\lambda\mu}^\nu = c_{\mu\lambda}^\nu$.

Remark 14.33. From now on, we will feel free to represent a Schubert class by its corresponding Young diagram. Take note of the following

- (1) The Young diagrams that represent Schubert classes for $\mathbb{G}_r\mathbb{P}^n$ must fit in an $(r+1) \times (n-r)$ box of unit squares. Thus, when computing products in the Chow ring using the formula $\{\lambda\} \cdot \{\mu\} = \sum_{\nu \in A^\bullet \mathbb{G}_r\mathbb{P}^n} c_{\lambda\mu}^\nu \{\nu\}$ are those ν that fit in that box.
- (2) The identity in $A^\bullet \mathbb{G}_r\mathbb{P}^n$ is $[\mathbb{G}_r\mathbb{P}^n]$ since intersecting a subvariety of $\mathbb{G}_r\mathbb{P}^n$ with the whole space does not change the variety. We have $[\mathbb{G}_r\mathbb{P}^n] = (n-r, n-r+1, \dots, n) = \{0, \dots, 0\} = \lambda$. Note that $|\lambda| = 0$, which agrees with the fact that the class has codimension 0.
- (3) The class of a single point in $\mathbb{G}_r\mathbb{P}^n$, i.e., of an r -plane in $\mathbb{P}^n$, has the form $a = (0, 1, \dots, r) = \{n-r, \dots, n-r\} = \lambda$, with Young diagram consisting of the $(r+1) \times (n-r)$ rectangle. Note that the codimension is $|\lambda| = (r+1)(n-r) = \dim \mathbb{G}_r\mathbb{P}^n$, as expected.

Example 14.34. The reader should verify the following multiplication table for $A^\bullet \mathbb{G}_1\mathbb{P}^3$:

*	1					
1	1					
		 + 				0
				0	0	0
			0		0	0
			0	0	0	0
		0	0	0	0	0

The class of  represents the condition “to meet a given line” and has codimension 1, therefore  represents the condition “to meet four given lines”. To determine how many lines in 3-space meet four generic lines, we use the table to compute:

$$\square^4 = \square^2 \left(\square\square + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) = \square \left(2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) = 2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}.$$

Since $\square$ is the class of a point, i.e., of a line in 3-space, we conclude that there are two lines meeting four generic lines in 3-space.

Here is a sketch of another proof that there are two lines meeting four general lines in 3-space. Call the four general lines L_1, L_2, L_3, L_4 . Walk along L_3 , and at each point $p \in L_3$, stop and look out at the lines L_1 and L_2 . They will appear to cross at some point. Draw a line from p through that point. One may check that the collection of all lines drawn in this way forms a quadric surface Q in 3-space—the zero-set of a single equation of degree two. Through each point in Q , there is a line lying on Q that meets L_1, L_2 and L_3 . Now consider line L_4 . A line and a surface in $\mathbb{P}^3$ must meet, and since L_4 is general, it will not be tangent to Q . Since Q is defined by an equation of degree two, the line will meet in two points, each of which corresponds to a line we drew earlier. These two lines are exactly those meeting L_1, L_2, L_3 , and L_4 .

Finally, we mention a proof given by Schubert using his “principle of conservation of number”: that the number of solutions will remain the same as the parameters of the configuration are continuously changed as long as the number of solutions stays finite. Suppose that L_1 and L_2 lie in a plane P_{12} and L_3 and L_4 lie in a plane P_{34} . How many lines meet these four lines now that they are in special position? Since we are working in projective space, both $L_1 \cap L_2$ and $L_3 \cap L_4$ will be points. The line through these two points yields one solution. The intersection of P_{12} and P_{34} will be a line. Since that line sits in P_{12} it will meet both L_1 and L_2 , and it similarly meets both L_3 and L_4 .

APPENDIX A. VECTOR CALCULUS IN EUCLIDEAN SPACES

A.1. Derivatives. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with the understanding that what we say below can be suitably modified to apply to the more general case where the domain and codomain of f are arbitrary open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Definition A.1. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open. A function $f: U \rightarrow V$ is *smooth* if each of its components has all partial derivatives of all orders. If f is bijective and has a smooth inverse, then we say that f is a *diffeomorphism*.

Definition A.2. The function f is *differentiable* at $p \in \mathbb{R}^n$ if there is a linear function $Df_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{h \rightarrow 0} \frac{|f(p+h) - f(p) - Df_p(h)|}{|h|} = 0.$$

The function Df_p is called the *derivative of f at p* . If each of the component functions of f have partial derivatives of all orders at p , then f is *smooth* at p , also known as being in the C^∞ , (in which case f is differentiable at p —see Theorem A.3.) We say f is *differentiable* (resp. *smooth*), if it is differentiable (resp. smooth) at all $p \in \mathbb{R}^n$.

Note: In this text will assume all of our differentiable mappings are smooth and use the words “differentiable” and “smooth” interchangeably.

Theorem A.3. If f is differentiable at $p \in \mathbb{R}^n$, then each partial derivative of each component function of f , i.e., $\partial f_i(p)/\partial x_i$, exists, and Df_p is the linear map associated with the Jacobian matrix of f at p :

$$Jf_p := \begin{bmatrix} \frac{\partial f_1(p)}{\partial x_1} & \cdots & \frac{\partial f_1(p)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(p)}{\partial x_1} & \cdots & \frac{\partial f_m(p)}{\partial x_n} \end{bmatrix}$$

Conversely, if each partial derivative of each component function exists and is continuous in an open set containing p , then f is differentiable at p .

If $m = 1$, then Jf_p is a single row vector called the *gradient of f at p* , denoted $\nabla_p f$ or $\text{grad}_p f$. If $n = 1$ and m is arbitrary, then f is a parametrized curve and Jf_p is a single column vector = the *velocity of the curve at p* . (In general, the columns of Jf_p may be thought of as *tangent vectors spanning the tangent space to f at p* .)

In what follows below, we make statements about functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with the understanding that, with minor adjustments, they apply equally well to functions of the form $f: U \rightarrow V$ for open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Theorem A.4. (Chain Rule.) If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at u , and $g: \mathbb{R}^m \rightarrow \mathbb{R}^p$ is differentiable at $f(u)$, then the composition $g \circ f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ is differentiable at u and

$$D(g \circ f)_u = Dg_{f(u)} \circ Df_u$$

or, in terms of Jacobian matrices,

$$J(g \circ f)_u = Jg_{f(u)} \circ Jf_u.$$

Theorem A.5. (*Inverse Function Theorem.*) Suppose that all the partial derivatives of each component function of $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ exist and are continuous in an open set containing u and $\det Jf_u \neq 0$. Then there is an open set W containing u and an open set V containing $f(u)$ such that $f: W \rightarrow V$ has a continuous inverse $f^{-1}: V \rightarrow W$ which is differentiable for all $v \in V$ and

$$J(f^{-1})_v = (Jf_{f^{-1}(v)})^{-1}.$$

Definition A.6. Let U, V be open subsets of $\mathbb{R}^n$. A mapping $f: U \rightarrow V$ is a *local diffeomorphism* if for all $p \in U$ there exists an open neighborhood $\tilde{U} \subseteq U$ of p such that $f(\tilde{U})$ is an open subset of V

$$f|_{\tilde{U}}: \tilde{U} \rightarrow f(\tilde{U})$$

is a diffeomorphism.

Remark A.7. Note that if $f: U \rightarrow V$ is a local diffeomorphism, the $f(U)$ is open in V . Further, by the inverse function theorem, if $\det(Jf_u) \neq 0$ for all $u \in U$, then f is a local diffeomorphism.

Definition A.8. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a differentiable function, and let $p \in \mathbb{R}^n$. The *best affine approximation to f at p* is the function

$$Af_p(x) := f(p) + Df_p(x).$$

A.2. Classical integral vector calculus.

A.2.1. Integral over a set.

Definition A.9. Let K be a bounded subset of $\mathbb{R}^n$, and let $f: K \rightarrow \mathbb{R}$ be a bounded function. To define the integral of f over K , choose a rectangle A containing K and define the function $\tilde{f}$ on A which agrees with f on K and is zero otherwise. The *integral* of f over K is then

$$\int_K f := \int_A \tilde{f},$$

provided this integral exists.

Remark A.10.

- (1) One may show that the definition does not depend on the choice of the bounding rectangle, A .
- (2) If $\int_K f$ exists, we say f is *integrable* (over K).
- (3) In class, we showed that if f is a bounded function defined on any rectangle A and is continuous except for a set of volume zero, then f is integrable over A . It follows that if K , from above, is compact (i.e., closed and bounded), has boundary of volume zero, and f is continuous on K , then f is integrable over K . Recall that the *boundary* of K is the set

$$\partial K := \overline{K} \cap \overline{K'},$$

the intersection of the closure of K and the closure of the complement of K .

Theorem A.11. (*Change of variables.*) Let $K \subset \mathbb{R}^n$ be a compact and connected set having boundary with measure zero. Let U be an open set containing K . Suppose that $\phi: U \rightarrow \mathbb{R}^n$ is a C^0 -mapping (i.e., with continuous first partials) and such that ϕ is injective on the interior K° and $\det(J\phi) \neq 0$ on K° . Then if $f: \phi(K) \rightarrow \mathbb{R}$ is continuous, it follows that

$$\int_{\phi(K)} f = \int_K (f \circ \phi) |\det(J\phi)|.$$

For the definitions of *boundary* and *interior*, see Definition B.1. In the change of variables theorem, we have a sequence of mappings

$$K \xrightarrow{\phi} \phi(K) \xrightarrow{f} \mathbb{R}^n.$$

The mapping ϕ is the change of variables. For instance, if we are integrating a function over a 2-sphere, K might be a rectangle and ϕ spherical coordinates.

A.2.2. Curve integrals.

Definition A.12. Let $C: [a, b] \rightarrow \mathbb{R}^n$ be a differentiable curve in n -space, and let f be a real-valued function defined on the image of C . The *path integral* of f along C is

$$\int_C f := \int_a^b (f \circ C) |C'|.$$

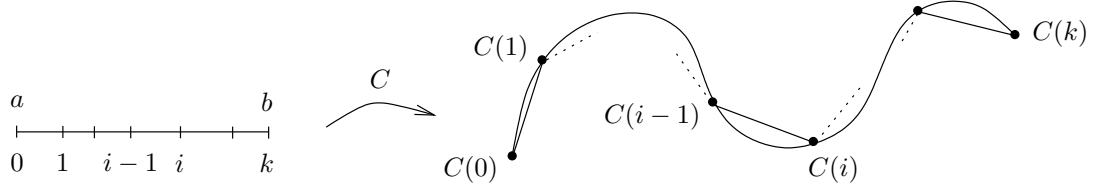
If $f = 1$, we get the *length* of C :

$$\text{length}(C) := \int_C 1 = \int_a^b |C'|.$$

Remark A.13.

- (1) In the above displayed formulas, the stuff on the left of the equals sign is just notation for the stuff on the right-hand side. In other words, the left-hand side has no independent meaning. The same holds for the formulas appearing in the definitions below.
- (2) The path integral is sometimes called the *line integral* or the *curve integral*. Another notation for the path integral is $\int_C f dC$.
- (3) The above integrals may not exist. They will exist if, for example, f is continuous on the image of C and C' is continuous.
- (4) Think of the factor of $|C'(t)|$ as a “stretching” factor, recording how much the domain, $[a, b]$ is stretched by C as it is placed in $\mathbb{R}^n$.

Geometric Motivation. The integral of f along C is meant to measure the weighted (by f) length of C . To see where the definition comes from, imagine how one would go about calculating the length of C . One might approximate the image of C by line segments, and in order to do this, it is natural to take a partition of the domain, $[a, b]$, say $a = t_0 \leq t_1 \leq \dots \leq t_k = b$. The image of these partition points can be used to break up the image of C into line segments as illustrated below:



Approximating a curve with matchsticks.

We want to add up the lengths of these line segments, weighting each one using f , which we might as well evaluate at an endpoint of each segment. Thus, we get

$$\begin{aligned}
 \text{weighted length of } C &\approx \sum_{i=1}^k f(C(t_i)) |C(t_i) - C(t_{i-1})| \\
 &= \sum_{i=1}^k f(C(t_i)) \left| \frac{C(t_i) - C(t_{i-1})}{t_i - t_{i-1}} \right| (t_i - t_{i-1}) \\
 &\approx \int_a^b (f \circ C) |C'(t)|.
 \end{aligned}$$

Definition A.14. Let $C: [a, b] \rightarrow \mathbb{R}^n$ be a curve in n -space, and let $F = (F_1, \dots, F_n)$ be a vector field in $\mathbb{R}^n$ defined on the image of C . The *flow* of F along C is

$$\int_C F \cdot dC := \int_C \omega_F,$$

where $\omega_F := F_1 dx_1 + \dots + F_n dx_n$ is the 1-form associated with F .

Remark A.15.

- (1) Another notation for the flow is $\int_C F \cdot \vec{t}$. If C is a closed curve, meaning $C(a) = C(b)$, then one often sees the notation $\oint_C F \cdot dC$.
- (2) Working out the definition given above, one sees that

$$\int_C F \cdot dC = \int_a^b F(C(t)) \cdot C'(t) dt.$$

Geometric Motivation. As claimed above, the flow can be expressed as $\int_a^b F(C(t)) \cdot C'(t) dt$, which can be re-written as

$$\int_a^b \left[F(C(t)) \cdot \frac{C'(t)}{|C'(t)|} \right] |C'(t)| dt.$$

Notice the unit tangent vector appearing inside the square brackets. The quantity inside the brackets is thus the component of F in the direction in which C is moving. The factor of $|C'(t)|$ is the stretching factor which appears when taking a path integral of a function. Comparing with the geometric motivation given for the path integral, we see that the flow is measuring the length of C , weighted by the component of F along C . Hence, it makes sense to call this integral the flow of F along C .

A.2.3. SURFACE INTEGRALS.

Definition A.16. Let $S: D \rightarrow \mathbb{R}^3$ with $D \subseteq \mathbb{R}^2$ be a differentiable surface in 3-space, and let f be a real-valued function defined on the image of S . The *surface integral* of f along S is

$$\int_S f := \int_D (f \circ S) |S_u \times S_v|,$$

where S_u and S_v are the partial derivatives of S with respect to u and v , respectively. If $f = 1$, we get the *surface area* of S :

$$\text{area}(S) = \int_S 1 = \int_D |S_u \times S_v|.$$

Remark A.17.

- (1) The cross product of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ in $\mathbb{R}^3$ is given by

$$v \times w := (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1),$$

which may be easier to remember with one of the following mnemonic devices:

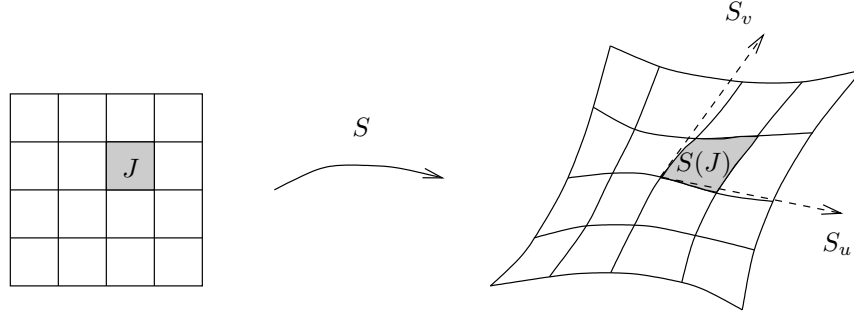
$$v \times w = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = ([2, 3], [3, 1], [1, 2]),$$

thinking of reading off the numbers 1, 2, and 3 arranged around a circle.

The important things to remember about the cross product are: (i) it is a vector perpendicular to the plane spanned by v and w , (ii) its direction is given by the “right-hand rule”, and (iii) its length is the area of the parallelogram spanned by v and w .

- (2) The factor $|S_u \times S_v|$ is the area of the parallelogram spanned by S_u and S_v and should be thought of as the factor by which space is locally stretched by f (analogous to $|C'(t)|$ for curve integrals).

Geometric Motivation. The integral of f along S is meant to measure the weighted (by f) surface area of S . To estimate this value, one might reasonably partition the domain of S . The image of the subrectangles would be warped parallelograms lying on the image of S :



Warped parallelograms on a surface.

The warped parallelograms can be approximated by scaling the parallelograms spanned by the partial derivative vectors, S_u and S_v . Hence, if J is a subrectangle of the partition, then $S(J)$ is a warped parallelogram whose area is approximately that of the parallelogram spanned by S_u and S_v , scaled by $\text{vol}(J)$. Recalling that the area of the parallelogram spanned by S_u and S_v is the length of their cross product, we get

$$\begin{aligned} \text{weighted surface area} &\approx \sum_J f(S(x_J)) \text{area}(S(J)) \quad (x_J \text{ any point in } J) \\ &\approx \sum_J f(S(x_J)) \text{area}(J) |S_u(x_J) \times S_v(x_J)| \\ &\approx \int_D (f \circ S) |S_u \times S_v|. \end{aligned}$$

Definition A.18. Let $D \subseteq \mathbb{R}^2$, and let $S: D \rightarrow \mathbb{R}^3$ be a differentiable surface in 3-space, and let $F = (F_1, F_2, F_3)$ be a vector field defined on the image of S . The *flux* of F through S is

$$\int_S F \cdot \vec{n} := \int_S \omega_F,$$

where

$$\begin{aligned} \omega_F &:= F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy \\ &= F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy, \end{aligned}$$

is the *flux 2-form* for F .

Remark A.19. (1) From the definition, it follows that

$$\int_S F \cdot \vec{n} = \int_D (F \circ S) \cdot (S_u \times S_v).$$

- (2) To calculate the flux of a vector field F through a hypersurface, S , in $\mathbb{R}^n$, integrate $\int_S \omega_F$ where $\omega_F := \sum_{i=1}^n (-1)^{i-1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. The notation signifies omitting dx_i and wedging all of the remaining dx_j 's in the i -term. By hypersurface, we mean that the domain of S sits in $\mathbb{R}^{n-1}$. Letting $n = 3$ recovers the usual notion of flux through a (2-dimensional) surface.

Geometric Motivation. To see the geometric motivation behind the definition of flux, one first needs to check that our definition for flux is equivalent to the integral $\int_D (F \circ S) \cdot (S_u \times S_v)$. Then, re-writing the integral gives:

$$\begin{aligned} \text{flux} &= \int_D (F \circ S) \cdot (S_u \times S_v) \\ &= \int_D \left[(F \circ S) \cdot \frac{S_u \times S_v}{|S_u \times S_v|} \right] |S_u \times S_v| \end{aligned}$$

Note the unit normal appearing inside the square brackets. Thus, the quantity inside the brackets is the component of F in the unit normal direction. The factor of $|S_u \times S_v|$ appearing outside the brackets is the “stretching” factor which appears in our earlier definition of the integral of a function along a surface. In light of the geometric motivation given in that situation (above), we see that the integral is measuring the surface area of S , weighted by the normal component of F normal to the surface.

A.2.4. SOLID INTEGRALS.

Definition A.20. Let $V: D \rightarrow \mathbb{R}^n$ with $D \subset \mathbb{R}^n$ be a differentiable mapping, and let f be a real-valued function defined on the image of V . The *integral of f over V* is

$$\int_V f := \int_D (f \circ V) |\det V'|,$$

where V' is the $n \times n$ Jacobian matrix for V . If $f = 1$, we get the *volume* of V :

$$\text{vol}(V) = \int_V 1 = \int_D |\det V'|.$$

Remark A.21.

- (1) The distinguishing feature for this integral is that both the domain and codomain are subsets of $\mathbb{R}^n$.
- (2) One could leave off the absolute value signs about the determinant. In that case, we would be taking orientation into account, and some parts of the integral could cancel with others.

Geometric Motivation. The integral is supposed to measure the volume of V weighted by f . To this end, partition the domain, D . Let J be a subrectangle of the partition. Then $V(J)$ is a warped rectangle which can be approximated by the parallelepiped spanned by the partials of V , i.e., the columns of the Jacobian matrix, V' , scaled by $\text{vol}(J)$. Recall that the volume of the parallelepiped spanned by the columns of a square matrix is given by the absolute value of the determinant of the matrix. Hence,

$$\begin{aligned} \text{weighted volume} &\approx \sum_J f(V(x_J)) \text{vol}(V(J)) \quad (x_J \text{ any point in } J) \\ &\approx \sum_J f(V(x_J)) |\det V'| \text{vol}(J) \\ &\approx \int_D (f \circ V) |\det V'|. \end{aligned}$$

A.2.5. *Grad, curl, and div.* Stokes' theorem says that

$$\int_C d\omega = \int_{\partial C} \omega,$$

for any k -chain C in $\mathbb{R}^n$ and $(k-1)$ -form ω in $\mathbb{R}^n$. In this section, we would like to consider the special cases: $k = 1, 2, 3$.

Case $k = 1$

To apply Stokes' theorem in the case $k = 1$, we start with a 1-chain C and a 0-form ω . We will consider the case where $C: [0, 1] \rightarrow \mathbb{R}^n$ is a curve in $\mathbb{R}^n$. Recall that a 0-form is a function: $\omega = f: \mathbb{R}^n \rightarrow \mathbb{R}$.

Definition A.22. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function. The vector field corresponding to the 1-form df is called the *gradient of f* :

$$\text{grad}(f) := \nabla f := (D_1 f, \dots, D_n f).$$

The function f (or sometimes $-f$) is called a *potential function* for the vector field.

In this case, Stokes' theorem says

$$\int_C df = \int_{\partial C} f.$$

The left-hand side is by definition the flow of ∇f along C , and the right-hand side is $f(C(1)) - f(C(0))$, the change in potential. We get the following classical result:

Theorem A.23. *The flow of the gradient vector field ∇f along C is given by the change in potential:*

$$\int_C \nabla f \cdot dC = f(C(1)) - f(C(0)).$$

Geometric interpretation of the gradient. To understand the gradient, let $p \in \mathbb{R}^n$, and let $v \in \mathbb{R}^n$ be a unit vector. Define

$$\begin{aligned} C_\varepsilon: [0, 1] &\rightarrow \mathbb{R}^n \\ t &\mapsto p + t\varepsilon v \end{aligned}$$

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$. If ε is small, then ∇f is approximately $\nabla f(p)$ along C_ε . Hence, the flow of ∇f along C_ε is approximately $\varepsilon \nabla f(p) \cdot v$ (noting that v is the unit tangent for C_ε). By Stokes' theorem, the flow is given by $f(C(1)) - f(C(0))$. Hence,

$$\nabla f(p) \cdot v \approx \frac{f(p + \varepsilon v) - f(p)}{\varepsilon}.$$

It turns out that taking the limit as $\varepsilon \rightarrow 0$ gives $\nabla f(p) \cdot v$ exactly. Therefore, the component of the gradient in any particular direction gives the rate of change of the function in that direction. Of course, the component is maximized when the direction points the same way as the gradient; so the gradient points in the direction of quickest increase of the function. In this way, the gradient can be thought of as “change density,” and Stokes' theorem says, roughly, that the integral of change density gives the total change, i.e., the change in potential.

Fundamental Theorem in one variable. Specializing further, let a, b be real numbers with $a < b$, and define $C(t) = a + t(b - a)$. Let $\omega = f: \mathbb{R} \rightarrow \mathbb{R}$ be a 0-form. Stokes' theorem says that $\int_C df = \int_{\partial C} f$. For the left-hand side, we get

$$\int_C df = \int_C f' dx = \int_0^1 (f' \circ C) C' = \int_a^b f',$$

using the change of variables theorem. Thus, Stokes' says

$$\int_a^b f' = \int_{\partial C} f = f(C(1)) - f(C(0)) = f(b) - f(a),$$

the fundamental theorem.

Case $k = 2$

To apply Stokes' theorem in the case $k = 2$, we start with a 2-chain and a 1-form. For the 2-chain, we will take a surface $S: D \rightarrow \mathbb{R}^3$, where $D = [0, 1] \times [0, 1]$. The 1-form looks like $\omega = F_1 dx + F_2 dy + F_3 dz$. Let $F := (F_1, F_2, F_3)$ be the corresponding vector field. For Stokes' theorem, we need to consider $d\omega$. A straightforward calculation (do it!) yields

$$d\omega = (D_2 F_3 - D_3 F_2) dy \wedge dz - (D_3 F_1 - D_1 F_3) dx \wedge dz + (D_1 F_2 - D_2 F_1) dx \wedge dy,$$

which corresponds to the vector field called the *curl* of F .

Definition A.24. Let $F = (F_1, F_2, F_3)$ be a vector field on $\mathbb{R}^3$. The *curl* of F is the vector field

$$\text{curl}(F) := \nabla \times F := \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ D_1 & D_2 & D_3 \\ F_1 & F_2 & F_3 \end{bmatrix} := (D_2 F_3 - D_3 F_2, D_3 F_1 - D_1 F_3, D_1 F_2 - D_2 F_1).$$

where D_i denotes the derivative with respect to the i -th variable.

Stokes' theorem says

$$\int_S d\omega = \int_{\partial S} \omega.$$

By definition, the left-hand side is the flux of the $\text{curl}(F)$ through S and the right-hand side is the flow of F along the boundary. We get the classical result:

Theorem A.25. *The flux of the curl of F through the surface S is equal to the flow of F along the boundary of the surface:*

$$\int_S \text{curl}(F) \cdot \vec{n} = \int_{\partial S} F \cdot dC$$

where $C = \partial S$.

Geometric interpretation of the curl. To get a physical understanding of the curl, pick a point $p \in \mathbb{R}^3$ and a unit vector v . Let D_ε be a parametrized disk of radius ε centered at p and lying in the plane normal to v . For ε small, the curl of F is approximately constant at $\text{curl}(F)(p)$ over D_ε . The flux of the curl is,

thus, approximately the normal component, $\text{curl}(F)(p) \cdot v$ times the area of D_ε . By Stokes' theorem, the flux is the circulation about the boundary. Thus,

$$\text{curl}(F)(p) \cdot v \approx \frac{1}{\text{area}(D_\varepsilon)} \int_C F \cdot dC,$$

where $C = \partial D_\varepsilon$. It turns out that we get equality if we take the limit as $\varepsilon \rightarrow 0$. Hence, the component of the curl in the direction of v measures the circulation of the original vector field about a point in the plane perpendicular to v . In this sense, the curl measures "circulation density". So roughly, Stokes' theorem says that integrating circulation density gives the total circulation.

Case $k = 3$

In this case, we are concerned with a 3-chain and a 2-form. We will consider the case where the chain is a solid $V: D \rightarrow \mathbb{R}^3$, where $D = [0, 1] \times [0, 1] \times [0, 1]$. We will assume that $\det V' \geq 0$ at all points. The 2-form can be written

$$\omega = F_1 dy \wedge dz - F_2 dx \wedge dz + F_3 dx \wedge dy.$$

A simple calculation (do it!) yields

$$d\omega = (D_1 F_1 + D_2 F_2 + D_3 F_3) dx \wedge dy \wedge dz.$$

Definition A.26. If F is a vector field on $\mathbb{R}^n$, the *divergence* of F is the scalar function

$$\text{div}(F) := \nabla \cdot F := \sum_{i=1}^n D_i F_i.$$

Stokes' theorem says that $\int_V d\omega = \int_{\partial V} \omega$. The left-hand side is

$$\begin{aligned} \int_V d\omega &= \int_V \text{div}(F) dx \wedge dy \wedge dz \\ &= \int_D \text{div}(F) \circ V \det(V') \\ &= \int_V \text{div}(F) \quad (\text{since we've assumed } \det(V') \geq 0). \end{aligned}$$

(The first equality follows from the definition of $\text{div}(F)$, the second from the definition of integration of a differential form, and the third from the definition of a solid integral, given earlier in this handout.) The right-hand side of Stokes' is by definition the flux of F through the boundary of V . The classical result is:

Definition A.27. The integral of the divergence of F over V is equal to the flux of F through the boundary of V :

$$\int_V \text{div}(F) = \int_{\partial V} F \cdot \vec{n}.$$

Geometric interpretation of the divergence. Pick a point $p \in \mathbb{R}^3$, and let V_ε be a solid ball of radius ε centered at p . If ε is small, the divergence of F will not

change much from $\operatorname{div}(F)(p)$ on V_ε . Hence, the integral of the divergence will be approximately $\operatorname{div}(F)(p)$ times the volume of V_ε . By Stokes' we get

$$\operatorname{div}(F)(p) \approx \frac{1}{\operatorname{vol}(V_\varepsilon)} \int_S F \cdot \vec{n},$$

where $S = \partial V$. Taking a limit gives an equality. Thus, the divergence measures “flux density”: the amount of flux per unit volume diverging from a given point. So Stokes' theorem in this case is saying that the integral of flux density gives the total flux.

APPENDIX B. TOPOLOGY

B.1. Topological spaces. This appendix is an extraction of the salient parts, for our notes, of [An outline summary of basic point set topology](#), by Peter May. (Note: this reference's definition of a *neighborhood* differs slightly from ours—we do not insist that a neighborhood is open. See below.)

Definition B.1. A *topology* on a set X is a collection of subsets τ of X such that

- (1) $\emptyset \in \tau$.
- (2) $X \in \tau$.
- (3) τ is closed under arbitrary unions.
- (4) τ is closed under finite intersections.

The elements of τ are called the *open sets* for the topology and (X, τ) (or just X , if τ is understood from context) is called a *topological space*. A *neighborhood* of a point $x \in X$ is any set N (not necessarily open) that contains an open set containing x . A subset of a topological space is *closed* if its complement is open. The *closure*, $\overline{A}$, of a subset $A \subseteq X$ is the intersection of all closed set containing A . The *interior* of A , denoted A° is the union of all open sets (in X) that are contained in A , and the *boundary* of A is the set $\partial A := A \setminus A^\circ$.

Definition B.2. A collection of subsets $\mathcal{B}$ of a set X is a *basis* for a topology on X if

- (1) $\mathcal{B}$ covers X : for each $x \in X$, there exists an element of $B \in \mathcal{B}$ containing x ;
- (2) if $x \in B' \cap B''$ for some $B', B'' \in \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$.

The topology *generated* by a basis $\mathcal{B}$ consists of all unions of sets in $\mathcal{B}$.

Exercise B.3. Let τ be the topology on X generated by a basis $\mathcal{B}$. Show that $U \in \tau$ if and only if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Example B.4. The *open ball* in $\mathbb{R}^n$ of radius $r \in \mathbb{R}_{\geq 0}$ centered at $x \in \mathbb{R}^n$ is the set

$$B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}.$$

Note that the empty set is an open ball (taking $r = 0$). The open balls generate the standard topology on $\mathbb{R}^n$.

In order to do integration on manifolds, the following concept is important:

Definition B.5. A topological space X is *second countable* if it has a countable basis, i.e., if it has a basis whose elements can be put in bijection with $\mathbb{N}$.

Definition B.6. A topology on a set X is *Hausdorff* if for all points $x \neq y$ in X , there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Exercise B.7. Let X be a Hausdorff topological space. Show that $\{x\}$ is closed for each point $x \in X$.

Definition B.8. Let X and Y be topological spaces. The *product topology* on $X \times Y$ has basis consisting of the sets $U \times V$ where U is open in X and V is open in Y . Unless otherwise indicated, we will always assume the product topology on $X \times Y$.

Exercise B.9. Show that a topological space X is Hausdorff if and only if the diagonal $\Delta := \{(x, x) \in X \times X : x \in X\}$ is closed.

Definition B.10. The *subspace topology* on a subset A of a topological space X is the set of intersections $A \cap U$ such that U is open in X . A subset of X endowed with the subspace topology is called a (topological) *subspace* of X .

Exercise B.11. Let X and Y be topological spaces with subsets $A \subseteq X$ and $B \subseteq Y$. Define two topologies on $A \times B$: on one hand, endow A and B with their respective subspace topologies, and then form the product topology on $A \times B$; and on the other hand, give $A \times B$ the subspace topology as a subset of $X \times Y$. Show these topologies are the same.

Definition B.12. Let X be a topological space, and let $\pi: X \rightarrow Y$ be a surjective function. The *quotient topology* on Y is the set of subsets $U \subseteq Y$ such that $\pi^{-1}(U)$ is open in X .

Exercise B.13. Show that a subspace of a Hausdorff space is Hausdorff and that the product of Hausdorff spaces is Hausdorff. Show that the quotient of a Hausdorff space need not be Hausdorff.

B.2. Continuous functions and homeomorphisms. Let $f: X \rightarrow Y$ be a function between topological spaces. Then $f: X \rightarrow Y$ is *continuous* if $f^{-1}(U)$ is open in X for each open set U in Y . To check continuity, it suffices to check that $f^{-1}(U)$ is open for each set U in a basis for Y , or to check that $f^{-1}(A)$ for each closed subset A of Y .

Exercise B.14.

- (1) Show that $f: X \rightarrow Y$ is continuous if and only if for each $x \in X$ and neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.
- (2) Show that a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous if and only if it satisfies the usual ϵ - δ definition of continuity.

If $f: X \rightarrow Y$ is bijective and both it and its inverse, f^{-1} are continuous, then f is a *homeomorphism*. In that case, we think of X and Y as being “isomorphic” or “the same” as topological spaces.

B.3. Connectedness. Let $I = [0, 1]$ be the unit interval in $\mathbb{R}$, and let X be a topological space. A *path* in X is a continuous function $f: I \rightarrow X$ connecting the points $f(0)$ and $f(1)$.

Definition B.15.

- (1) X is *connected* if the only subsets of X that are both open and closed are $\emptyset$ and X , itself.
- (2) X is *path connected* if every two points of X can be connected by a path.

Exercise B.16. Let $h: X \rightarrow Y$ be a continuous mapping, and let $A \subseteq X$ be a connected subspace of X . Show that $h(A)$ is connected.

Definition B.17. Define two equivalence relations on a topological space X . First, say $x \sim y$ for two points in X if x and y are both elements in some connected subspace of X . Second, say $x \approx y$ if there is a path in X from x to y . An equivalence class for $\sim$ is a *connected component* of X , and an equivalence class for $\approx$ is a *path component* of X .

If x, y are in some path component of a topological space X , let f be a path in X joining x to y . Then the image of f , with the subspace topology, is connected. Hence, each connected component is the union of path components. It is not necessarily true that the set of connected components is the same as the set of path components, but we will see below (Theorem B.19) that in the case of relevance to our study—locally Euclidean spaces—the two notions coincide.

Definition B.18. Let X be a topological space.

- (1) X is *locally connected* if for each $x \in X$ and each neighborhood U of x , there exists a connected neighborhood V of x contained in U .
- (2) X is *locally path connected* if for each $x \in X$ and each neighborhood U of x , there exists a path connected neighborhood V of x contained in U .

Theorem B.19. *If a topological space is locally path connected, then its components and path components coincide.*

B.4. Compactness.

Definition B.20. A topological space X is *compact* if every open cover of X has a finite subcover, that is: if $\{U_\alpha\}_\alpha$ is a family of open subsets of X such that $X = \cup_\alpha U_\alpha$, then there are finitely many indices $\alpha_1, \dots, \alpha_k$ such that $X = \cup_{i=1}^k U_{\alpha_i}$.

Here is a list of first properties of compactness:

- A subspace A of a topological space X is compact if and only if every open cover of A by open subsets of X has a finite subcover.
- A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
- A compact subspace of a compact Hausdorff space is closed.
- The continuous image of a compact space is compact.
- If $f: X \rightarrow Y$ is a bijective continuous mapping, X is compact, and Y is Hausdorff, then f is a homeomorphism, i.e., it has a continuous inverse.

B.5. Partition of unity. Let M be a manifold that is second countable and Hausdorff, and let $\mathcal{U}$ be an open cover of M . Then there exists a collection Ξ of smooth functions $M \rightarrow [0, 1] \subset \mathbb{R}$ such that:

- For each $\lambda \in \Xi$ there exist $U \in \mathcal{U}$ containing $\text{supp}(\lambda)$. [Note: The *support* of λ is defined by

$$\text{supp}(\lambda) := \{x \in M : \lambda(x) \neq 0\}.$$

- For each $x \in M$, there exists an open neighborhood V of x such that $\lambda|_V = 0$ for all but finitely many $\lambda \in \Xi$, and

$$\sum_{\lambda \in \Xi} \lambda(x) = 1.$$

[Note: the sum makes sense since all but finitely terms are zero.]

Partitions of unity are important to glue together locally-defined objects. For instance, integration of forms on manifolds is at first defined locally, then glued together for a global definition. For more on the existence of partitions of unity, see the Wikipedia page on [paracompact spaces](#).

APPENDIX C. MEASURE THEORY

For more details on measure theory, see *The Elements of Integration and Lebesgue Measure*, by Bartle, [1], or [Math 321, Real Analysis](#), by Perkinson.

To put a “measure” on a set, we first need to divvy up the set into measurable pieces:

Definition C.1. A σ -algebra (sigma algebra) on a set X is a collection of subsets $\Sigma \subseteq 2^X$ satisfying:

- (1) $\emptyset \in \Sigma$ and $X \in \Sigma$,
- (2) (closed under complementation) $A \in \Sigma$ implies $A^c \in \Sigma$,
- (3) (closed under countable unions) $\{A_i\}_{i \in \mathbb{N}} \subseteq \Sigma$ implies $\cup_{i \in \mathbb{N}} A_i \in \Sigma$.

Next, we decide the size of each piece:

Definition C.2. Let X be a set with σ -algebra Σ . A *measure* on (X, σ) is a function

$$\mu: \Sigma \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

such that

- (1) $\mu(\emptyset) = 0$,
- (2) (additive) $\mu(\cup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$ for pairwise disjoint $A_i \in \Sigma$.

The triple (X, σ, μ) is called a *measure space*.

We can then define integration of nice functions on the set. We first define “nice”:

Definition C.3. Let (X, σ, μ) be a measure space. A function $f: X \rightarrow \mathbb{R}$ is *measurable* if

$$f^{-1}((a, \infty)) := \{x \in X : f(x) > a\} \in \Sigma$$

for all $a \in \mathbb{R}$.

Then define the integral of a simple function:

Definition C.4. The *characteristic function* of a subset $A \subseteq X$ is the function defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A, \end{cases}$$

If A is measurable and $c \in \mathbb{R}$, then

$$\int c \chi_A := c \mu(A).$$

A function $\phi: X \rightarrow \mathbb{R}$ is *simple* if it is a linear combination of characteristic functions: $\phi = \sum_{i=1}^k c_i \chi_{E_i}$ for some measurable sets $E_1, \dots, E_k$. In that case, its *integral* is

$$\int \phi := \sum_{i=1}^k \int c_i \chi_{E_i} = \sum_{i=1}^k c_i \mu(E_i).$$

Finally, we define the integral of a measurable function by approximating it with simple functions:

Definition C.5. Let f be a measurable function on a measure space (X, σ, μ) . If $f(x) \geq 0$ for all $x \in X$, define

$$\int f := \sup_{\phi} \left\{ \int \phi \right\}$$

where the sup is over all simple functions ϕ such that $0 \leq \phi(x) \leq f(x)$ for all $x \in X$ (see Figure 35).

If f is not necessarily nonnegative, then write $f = f^+ - f^-$ where

$$f^+(x) := \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f^-(x) := \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then f is *integrable* if $\int f^+$ and $\int f^-$ are finite, in which case

$$\int f := \int f^+ - \int f^-.$$

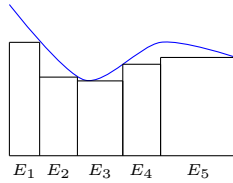


FIGURE 35. Approximating a function with a simple function.

Lebesgue measure. We now turn to the case of interest for this text: the standard measure on $\mathbb{R}^n$. To approximate the size of a set $X \subset \mathbb{R}^n$, we cover it with rectangles:

Definition C.6. If $I \in \mathbb{R}^n$ is a rectangle, define $\mu(I)$ to be the product of the lengths of its sides. The *outer measure* of $X \subset \mathbb{R}^n$ is

$$\text{outer}(X) = \inf \left\{ \sum_{i=1}^{\infty} \mu(I_i) \right\}$$

where the inf is over all sequences of rectangles $I_1, I_2, \dots$ covering X , i.e., such that $X \subseteq \bigcup_{i=1}^{\infty} I_i$. The set X is *Lebesgue measurable* if it “splits additively in measure”:

$$\text{outer}(X) = \text{outer}(X \cap A) + \text{outer}(X \cap A^c)$$

for all $A \subseteq \mathbb{R}^n$.

Theorem C.7. The collection of Lebesgue measurable sets $\mathcal{L}$ in $\mathbb{R}^n$ forms a σ -algebra. It contains all open sets (and, hence, all closed sets). Outer measure restricted to $\mathcal{L}$ forms a measure, i.e., defining $\mu(A) = \text{outer}(A)$ for all $A \in \mathcal{L}$, it follows that $(\mathbb{R}^n, \mathcal{L}, \mu)$ is a measure space.

APPENDIX D. SIMPLICIAL HOMOLOGY

D.1. Exact sequences. A sequence of linear mappings of vector spaces

$$V' \xrightarrow{f} V \xrightarrow{g} V''$$

is *exact* (or *exact at V*) if $\text{im}(f) = \ker(g)$. A *short exact sequence* of vector spaces is a sequence of linear mappings

$$0 \rightarrow V' \xrightarrow{f} V \xrightarrow{g} V'' \rightarrow 0$$

exact at V' , V , and V'' .

Exercise D.1. For a short exact sequence of R -modules as above, show that

- (1) f is injective;
- (2) g is surjective;
- (3) V'' is isomorphic to $\text{coker}(f)$;
- (4) $\dim V = \dim V' + \dim V''$.

In general, a sequence of linear mappings

$$\dots \rightarrow V_i \rightarrow V_{i+1} \rightarrow \dots$$

is *exact* if it is exact at each V_i (except the first and last, if they exist).

Consider a commutative diagram of linear mappings with exact rows

$$\begin{array}{ccccccc} V' & \xrightarrow{f} & V & \xrightarrow{g} & V'' & \longrightarrow & 0 \\ \downarrow \phi' & & \downarrow \phi & & \downarrow \phi'' & & \\ 0 & \longrightarrow & W' & \xrightarrow{h} & W & \xrightarrow{k} & W'' \end{array} .$$

(By *commutative*, we mean $\phi \circ f = h \circ \phi'$ and $\phi'' \circ g = k \circ \phi$.)

The *snake lemma* says there is an exact sequence

$$\ker \phi' \rightarrow \ker \phi \rightarrow \ker \phi'' \rightarrow \text{coker } \phi' \rightarrow \text{coker } \phi \rightarrow \text{coker } \phi'' .$$

If f is injective, then so is $\ker \phi' \rightarrow \ker \phi$, and if k is surjective, so is $\operatorname{coker} \phi \rightarrow \operatorname{coker} \phi''$.

Exercise D.2. Prove the snake lemma.

A *chain complex* C of R -modules is a collection of R -modules $\{C_n\}_n$ together with R -module maps $\{d_n: C_n \rightarrow C_{n-1}\}_n$ such that $d_{n-1} \circ d_n = 0$, in other words, $\operatorname{im}(d_n) \subseteq \ker(d_{n-1})$. Sometimes for convenience, subscripts for d are often omitted and we simply write $d^2 = 0$.

Given chain complexes C and D , a chain map f from C to D is a collection of R -module maps $\{f_n: C_n \rightarrow D_n\}$ satisfying $fd = df$, namely, the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} \xrightarrow{d} \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} \xrightarrow{d} \cdots \end{array}$$

A short exact sequence of chain complexes of R -modules is a sequence of chain maps $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ such that at each level n , we have a short exact sequence $0 \rightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \rightarrow 0$.

The n -th *homology group* is defined to be the quotient $H_n(C) := \ker(d_n)/\operatorname{im}(d_{n+1})$. The homology groups measures how far away the chain complex is from being exact. If all of the homology groups are zero, then we say that the chain complex is *exact*.

Theorem D.3. A short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of chain complexes induces the connecting homomorphisms $\partial: H_n(C) \rightarrow H_{n-1}(A)$ and a long exact sequence of homology groups:

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

Proof. We use the notation $d_n^A := d_n: A_n \rightarrow A_{n-1}$, and similarly for B and C . There are induced mappings Applying the snake lemma to

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f} & B_n & \xrightarrow{g} & C_n \longrightarrow 0 \\ & & \downarrow d_n^A & & \downarrow d_n^B & & \downarrow d_n^C \\ 0 & \longrightarrow & A_{n+1} & \xrightarrow{f} & B_{n+1} & \xrightarrow{g} & C_{n+1} \xrightarrow{d} 0 \end{array}$$

to get the exact sequence

$$0 \rightarrow \ker(d_n^A) \rightarrow \ker(d_n^B) \rightarrow \ker(d_n^C) \rightarrow \operatorname{coker}(d_n^A) \rightarrow \operatorname{coker}(d_n^B) \rightarrow \operatorname{coker}(d_n^C) \rightarrow 0.$$

Next, define the mapping

$$\begin{aligned} \delta_n^A: \operatorname{coker}(d_{n+1}) &= A_n/\operatorname{im}(d_{n+1}) \rightarrow \ker(d_{n-1}) \\ [a] &\mapsto d_n^A(a). \end{aligned}$$

Then $\ker \delta_n^A = \ker(d_n)/\operatorname{im}(d_{n+1}) = H_n(A)$ and $\operatorname{coker} \delta_n^A = \ker(d_{n-1})/\operatorname{im}(d_n) = H_{n-1}(A)$, and similarly for B and C . By our previous exact sequence, we see the following diagram has exact rows:

$$\begin{array}{ccccccc}
\operatorname{coker}_{n+1}(d_{n+1}^A) & \longrightarrow & \operatorname{coker}_{n+1}(d_{n+1}^B) & \xrightarrow{g} & \operatorname{coker}_{n+1}^C & \longrightarrow & 0 \\
\downarrow \delta_n^A & & \downarrow \delta_n^B & & \downarrow \delta_n^C & & \\
0 & \longrightarrow & \ker_{n+1}(d_{n-1}^A) & \longrightarrow & \ker_{n+1}(d_{n-1}^B) & \longrightarrow & \ker_{n+1}(d_{n-1}^C).
\end{array}$$

The result now follows by applying the snake lemma to the above diagram. $\square$

D.2. Simplicial complexes.

D.2.1. First definitions. An (abstract) *simplicial complex* Δ on a finite set S is a collection of subsets of S , closed under the operation of taking subsets. The elements of a simplicial complex Δ are called *faces*. An element $\sigma \in \Delta$ of cardinality $i + 1$ is called an *i -dimensional face* or an *i -face* of Δ . The empty set, $\emptyset$, is the unique face of dimension -1 . Faces of dimension 0, i.e., elements of S , are *vertices* and faces of dimension 1 are *edges*.

The maximal faces under inclusion are called *facets*. To describe a simplicial complex, it is often convenient to simply list its facets—the other faces are exactly determined as subsets. The *dimension* of Δ , denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces. A simplicial complex is *pure* if each of its facets has dimension $\dim(\Delta)$.

Example D.4. If $G = (V, E)$ is a simple connected graph (undirected with no multiple edges or loops), then G is the pure one-dimensional simplicial complex on V with E as its set of facets.

Example D.5. Figure 36 pictures a simplicial complex Δ on the set $[5] := \{1, 2, 3, 4, 5\}$:

$$\Delta := \{\emptyset, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{12}, \bar{13}, \bar{23}, \bar{24}, \bar{34}, \bar{123}\},$$

writing, for instance, $\bar{23}$ to represent the set $\{2, 3\}$.

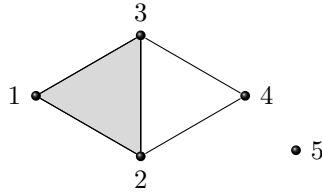


FIGURE 36. A 2-dimensional simplicial complex, Δ .

The sets of *faces* of each dimension are:

$$\begin{aligned}
F_{-1} &= \{\emptyset\} & F_0 &= \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} \\
F_1 &= \{\bar{12}, \bar{13}, \bar{23}, \bar{24}, \bar{34}\} & F_2 &= \{\bar{123}\}.
\end{aligned}$$

Its *facets* are $\bar{5}$, $\bar{24}$, $\bar{34}$, and $\bar{123}$. The dimension of Δ is 2, as determined by the facet $\bar{123}$. Since not all of the facets have the same dimension, Δ is not *pure*.

D.2.2. Simplicial homology. Let Δ be an arbitrary simplicial complex. By relabeling, if necessary, assume its vertices are $[n] := \{1, \dots, n\}$. For each i , let $F_i(\Delta)$ be the set of faces of dimension i , and define the *group of i -chains* to be the $\mathbb{R}$ -vector space with basis $F_i(\Delta)$:

$$C_i = C_i(\Delta) := \mathbb{R}F_i(\Delta) := \left\{ \sum_{\sigma \in F_i(\Delta)} a_\sigma \sigma : a_\sigma \in \mathbb{R} \right\}.$$

The *boundary* of $\sigma \in F_i(\Delta)$ is

$$\partial_i(\sigma) := \sum_{j \in \sigma} \text{sign}(j, \sigma) (\sigma \setminus j),$$

where $\text{sign}(j, \sigma) = (-1)^{k-1}$ if j is the k -th element of σ when the elements of σ are listed in order, and $\sigma \setminus j := \sigma \setminus \{j\}$. Extending linearly gives the *i -th boundary mapping*,

$$\partial_i : C_i(\Delta) \rightarrow C_{i-1}(\Delta).$$

If $i > n - 1$ or $i < -1$, then $C_i(\Delta) := 0$, and we define $\partial_i := 0$. We sometimes simply write ∂ for ∂_i if the dimension i is clear from context.

Example D.6. Suppose $\sigma = \{1, 3, 4\} = \overline{134} \in \Delta$. Then $\sigma \in F_2(\Delta)$, and

$$\text{sign}(1, \sigma) = 1, \quad \text{sign}(3, \sigma) = -1, \quad \text{sign}(4, \sigma) = 1.$$

Therefore,

$$\partial(\sigma) = \partial_2(\overline{134}) = \overline{34} - \overline{14} + \overline{13}.$$

The (*augmented*) *chain complex* of Δ is the complex

$$0 \longrightarrow C_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \longrightarrow 0.$$

The word *complex* here refers to the fact that $\partial^2 := \partial \circ \partial = 0$, i.e., for each i , we have $\partial_{i-1} \circ \partial_i = 0$.

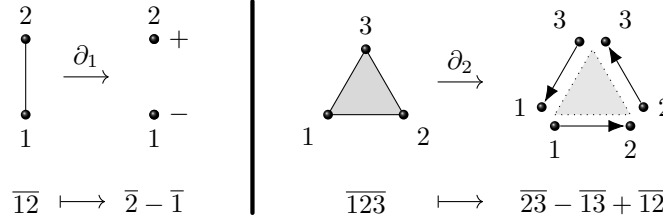


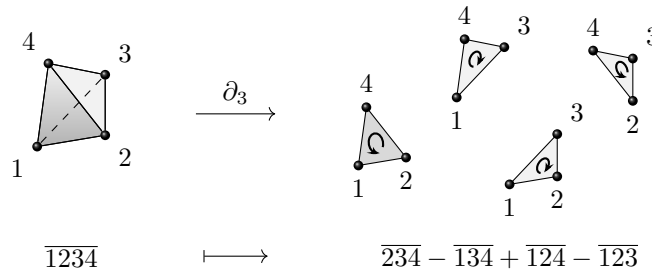
FIGURE 37. Two boundary mapping examples. Notation: if $i < j$, then we write $i \longrightarrow j$ for $\overline{ij}$ and $i \longleftarrow j$ for $-\overline{ij}$.

Figure 37 gives two examples of the application of a boundary mapping. Note that

$$\partial^2(\overline{12}) = \partial_0(\partial_1(\overline{12})) = \partial_0(\overline{2} - \overline{1}) = \emptyset - \emptyset = 0.$$

The reader is invited to verify $\partial^2(\overline{123}) = 0$.

Figure 38 shows the boundary of $\sigma = \overline{1234}$, the solid tetrahedron. Figure 39 helps to visualize the fact that $\partial^2(\sigma) = 0$. The orientations of the triangles may be thought of as inducing a “flow” along the edges of the triangles. These flows


 FIGURE 38. ∂_3 for a solid tetrahedron. Notation: if $i < j < k$,

then we write $\triangle_{i,j,k}$ for $\overline{ijk}$ and $\triangle_{i,j,k}$ for $-\overline{ijk}$.

cancel to give a net flow of 0. This should remind you of Stokes' theorem from multivariable calculus.

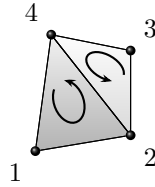


FIGURE 39. As seen in Figure 38, the boundary of a solid tetrahedron consists of oriented triangular facets.

Example D.7. Let Δ be the simplicial complex on $[4]$ with facets $\overline{12}$, $\overline{3}$, and $\overline{4}$ pictured in Figure 40. The faces of each dimension are:

$$F_{-1}(\Delta) = \{\emptyset\}, \quad F_0(\Delta) = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}, \quad F_1(\Delta) = \{\overline{12}\}.$$

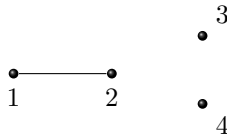


FIGURE 40. Simplicial complex for Example D.7.

Here is the chain complex for Δ :

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_1(\Delta) & \xrightarrow{\partial_1} & C_0(\Delta) & \xrightarrow{\partial_0} & C_{-1}(\Delta) \longrightarrow 0. \\
& & & & \overline{12} \longmapsto \overline{2} - \overline{1} & & \begin{array}{c} \overline{1} \\ \overline{2} \\ \overline{3} \\ \overline{4} \end{array} \begin{array}{c} \swarrow \\ \swarrow \\ \swarrow \\ \swarrow \end{array} \longrightarrow \emptyset
\end{array}$$

In terms of matrices, the chain complex is given by

$$\begin{array}{ccccccc}
& & \overline{12} & & & & \\
& & \overline{1} & \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} & & & \\
& & \overline{2} & & & & \\
& & \overline{3} & & & & \\
& & \overline{4} & & & & \\
0 & \longrightarrow & \mathbb{R} & \xrightarrow{\partial_1} & \mathbb{R}^4 & \xrightarrow{\partial_0} & \mathbb{R} \longrightarrow 0. \\
& & & & \begin{array}{c} \overline{1} \ \overline{2} \ \overline{3} \ \overline{4} \\ \emptyset \ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array} & &
\end{array}$$

The sequence is not exact since $\dim(\operatorname{im} \partial_1) = \dim \partial_1 = 1$, whereas by rank-nullity, $\dim(\ker(\partial_0)) = 4 - \dim \partial_0 = 3$.

Definition D.8. For $i \in \mathbb{Z}$, the i -th (reduced) homology of Δ is the vector space

$$\tilde{H}_i(\Delta) := \ker \partial_i / \operatorname{im} \partial_{i+1}.$$

In particular, $\tilde{H}_{n-1}(\Delta) = \ker(\partial_{n-1})$, and $\tilde{H}_i(\Delta) = 0$ for $i > n-1$ or $i < 0$. Elements of $\ker \partial_i$ are called i -cycles and elements of $\operatorname{im} \partial_{i+1}$ are called i -boundaries. The i -th (reduced) Betti number of Δ is the dimension of the i -th homology:

$$\tilde{\beta}_i(\Delta) := \dim \tilde{H}_i(\Delta) = \dim(\ker \partial_i) - \dim(\partial_{i+1}).$$

Remark D.9. To define ordinary (non-reduced) homology groups, $H_i(\Delta)$, and Betti numbers $\beta_i(\Delta)$, modify the chain complex by replacing $C_{-1}(\Delta)$ with 0 and ∂_0 with the zero mapping. The difference between homology and reduced homology is that $H_0(\Delta) \simeq \mathbb{R} \oplus \tilde{H}_0(\Delta)$ and, thus, $\beta_0(\Delta) = \tilde{\beta}_0(\Delta) + 1$. All other homology groups and Betti numbers coincide. From now on, we use “homology” to mean reduced homology.

In general, homology can be thought of as a measure of how close the chain complex is to being exact. In particular, $\tilde{H}_i(\Delta) = 0$ for all i if and only if the chain complex for Δ is exact. For the next several examples, we will explore how exactness relates to the topology of Δ .

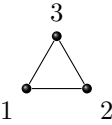
The 0-th homology group measures “connectedness”. Write $i \sim j$ for vertices i and j in a simplicial complex Δ if $\overline{ij} \in \Delta$. An equivalence class under the transitive closure of $\sim$ is a *connected component* of Δ .

Exercise D.10. Show that $\tilde{\beta}_0(\Delta)$ is one less than the number of connected components of Δ .

For instance, for the simplicial complex Δ in Example D.7,

$$\tilde{\beta}_0(\Delta) = \dim \tilde{H}_0(\Delta) = \dim(\ker \partial_0) - \dim(\partial_1) = 3 - 1 = 2.$$

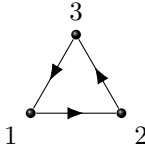
Example D.11. The hollow triangle,

$$\Delta = \{\emptyset, \bar{1}, \bar{2}, \bar{3}, \bar{12}, \bar{13}, \bar{23}\}$$


has chain complex

$$\begin{array}{c}
 \begin{array}{ccc}
 \bar{12} & \bar{13} & \bar{23} \\
 \bar{1} & \begin{pmatrix} -1 & -1 & 0 \end{pmatrix} \\
 \bar{2} & \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \\
 \bar{3} & \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}
 \end{array} & \begin{array}{c} \bar{1} \ \bar{2} \ \bar{3} \\ \emptyset \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{array} \\
 0 \longrightarrow \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^3 \xrightarrow{\partial_0} \mathbb{R} \longrightarrow 0.
 \end{array}$$

It is easy to see that $\dim(\partial_1) = \dim(\ker \partial_0) = 2$. It follows that $\tilde{\beta}_0(\Delta) = 0$, which could have been anticipated since Δ is connected. Since $\dim(\partial_1) = 2$, rank-nullity says $\dim(\ker \partial_1) = 1$, whereas $\partial_2 = 0$. Therefore, $\tilde{\beta}_1(\Delta) = \dim(\ker \partial_1) - \dim(\partial_2) = 1$. In fact, $\tilde{H}_1(\Delta)$ is generated by the 1-cycle

$$\bar{23} - \bar{13} + \bar{12} =$$


If we would add $\bar{123}$ to Δ to get a solid triangle, then the above cycle would be a boundary, and there would be no homology in any dimension. Similarly, a solid tetrahedron has no homology, and a hollow tetrahedron has homology only in dimension 2 (of rank 1).

Exercise D.12. Compute the Betti numbers for the simplicial complex formed by gluing two (hollow) triangles along an edge. Describe generators for the homology.

Example D.13. Consider the simplicial complex pictured in Figure 41 with facets $\bar{14}, \bar{24}, \bar{34}, \bar{123}$. It consists of a solid triangular base whose vertices are connected by edges to the vertex 4. The three triangular walls incident on the base are hollow.

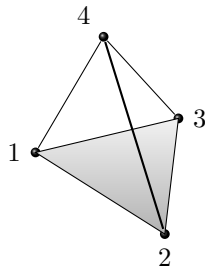
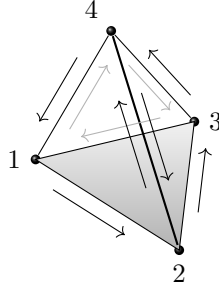


FIGURE 41. Simplicial complex for Example D.13.

What are the Betti numbers? The chain complex is:

$$\begin{array}{c}
\begin{array}{c} \overline{123} \\ \overline{12} \\ \overline{13} \\ \overline{14} \\ \overline{23} \\ \overline{24} \\ \overline{34} \end{array} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{array}{c} \overline{12} \ \overline{13} \ \overline{14} \ \overline{23} \ \overline{24} \ \overline{34} \\ \overline{1} \\ \overline{2} \\ \overline{3} \\ \overline{4} \end{array} \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\
\begin{array}{c} \overline{1} \ \overline{2} \ \overline{3} \ \overline{4} \\ \emptyset \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \end{array}
\end{array}
\begin{array}{c}
0 \rightarrow \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^6 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{\partial_0} \mathbb{R} \rightarrow 0.
\end{array}$$

By inspection, $\dim(\partial_2) = 1$ and $\dim(\partial_1) = \dim(\ker \partial_0) = 3$. Rank-nullity gives $\dim(\ker \partial_1) = 6 - 3 = 3$. Therefore, $\tilde{\beta}_0 = \tilde{\beta}_2 = 0$ and $\tilde{\beta}_1 = 2$. It is not surprising that $\tilde{\beta}_0 = 0$, since Δ is connected. Also, the fact that $\tilde{\beta}_2 = 0$ is easy to see since $\overline{123}$ is the only face of dimension 2, and its boundary is not zero. Seeing that $\tilde{\beta}_1 = 2$ is a little harder. Given the cycles corresponding to the three hollow triangles incident on vertex 4, one might suppose $\tilde{\beta}_1 = 3$. However, as conveyed in Figure 42, those cycles are not independent: if properly oriented their sum is the boundary of the solid triangle, $\overline{123}$; hence, their sum is 0 in the first homology group.



$$(\overline{12} + \overline{24} - \overline{14}) + (\overline{23} + \overline{34} - \overline{24}) + (\overline{14} - \overline{34} - \overline{13}) = \underbrace{(\overline{12} + \overline{23} - \overline{13})}_{\partial_2(\overline{123})}$$

FIGURE 42. A tetrahedron with solid base and hollow walls. Cycles around the walls sum to the boundary of the base, illustrating a dependence among the cycles in the first homology group.

D.2.3. A quick aside on algebraic topology. Algebraic topology seeks an assignment of the form $X \mapsto \alpha(X)$ where X is a topological space and $\alpha(X)$ is some algebraic invariant (a group, ring, etc.). If $X \simeq Y$ as topological spaces, i.e., if X and Y are homeomorphic, then we should have $\alpha(X) \simeq \alpha(Y)$ as algebraic objects—this is what it means to be *invariant*. The simplicial homology we have developed provides the tool for creating one such invariant.

Let X be a 2-torus—the surface of a donut. Draw triangles on the surface so that neighboring triangles meet vertex-to-vertex or edge-to-edge. The triangulation is naturally interpreted as a simplicial complex Δ . An amazing fact, of fundamental importance, is that the associated homology groups do not depend on the choice

of triangulation! In this way, we get an assignment

$$X \mapsto \tilde{H}_i(X) := \tilde{H}_i(\Delta),$$

and, hence, also $X \mapsto \tilde{\beta}_i(X) := \tilde{\beta}_i(\Delta)$, for all i .

In a course on algebraic topology, one learns that these homology groups do not see certain aspects of a space. For instance, they do not change under certain contraction operations. A line segment can be continuously morphed into a single point, and the same goes for a solid triangle or tetrahedron. So these spaces all have the homology of a point—in other words: none at all (all homology groups are trivial). A tree is similarly contractible to a point, so the addition of a tree to a space has no effect on homology. Imagine the tent with missing walls depicted in Figure 41. Contracting the base to a point leaves two vertices connected by three line segments. Contracting one of these line segments produces two loops meeting at a single vertex. No further significant contraction is possible—we are not allowed to contract around “holes” (of any dimension). These two loops account for $\tilde{\beta}_1 = 2$ in our previous calculation. As another example, imagine a hollow tetrahedron. Contracting a facet yields a surface that is essentially a sphere with three longitudinal lines connecting its poles, thus dividing the sphere into 3 regions. Contracting two of these regions results in a sphere—a bubble—with a single vertex drawn on it. No further collapse is possible. This bubble accounts for the fact that $\tilde{\beta}_2 = 1$ is the only nonzero Betti number for the sphere. (Exercise: verify that $\tilde{\beta}_2 = 1$ in this case.)

D.2.4. More examples. Let $\Delta \subset 2^{[n]}$ be a d -dimensional simplicial complex. For each $i \in \mathbb{Z}$ we have the space of i -dimensional chains $C_i := \mathbb{R}F_i$, the vector space of formal sums of i -dimensional faces of Δ . We have $C_i = 0$ for $i < -1$ and $i > d$, and since $F_{-1} = \{\emptyset\}$, we have $C_{-1} = \mathbb{R}$. The boundary mapping $\partial_i: C_i \rightarrow C_{i-1}$ is defined as follows: if $\sigma = \overline{\sigma_1 \dots \sigma_{i+1}} \in F_i$ where the σ_k are the vertices of σ and $\sigma_1 < \dots < \sigma_{i+1}$, then

$$\begin{aligned} \partial_i(\sigma) &= \sum_{k=1}^{i+1} (-1)^{k-1} \overline{\sigma_1 \dots \widehat{\sigma_k} \dots \sigma_{i+1}} \\ &= \overline{\sigma_2 \sigma_3 \sigma_4 \dots \sigma_{i+1}} - \overline{\sigma_1 \sigma_3 \sigma_4 \dots \sigma_{i+1}} + \overline{\sigma_1 \sigma_2 \sigma_4 \dots \sigma_{i+1}} + \dots \end{aligned}$$

for $-1 \leq i \leq d$ and $\partial_i = 0$, otherwise. Recall the definitions of the reduced homology groups and Betti numbers:

$$\tilde{H}_i := \tilde{H}_i(\Delta) := \ker \partial_i / \operatorname{im} \partial_{i+1}$$

$$\tilde{\beta}_i := \tilde{\beta}_i(\Delta) := \dim \tilde{H}_i = \operatorname{nullity}(\partial_i) - \dim(\partial_{i+1}).$$

Examples D.14.

I.1.

$$\begin{array}{cccc} \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

$$0 \rightarrow \mathbb{R}^4 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{R} \rightarrow 0$$

We have $\dim(\partial_0) = 1$, and hence by the rank-nullity theorem, $\text{nullity} = 4 - 1 = 3$. The only non-vanishing homology group is

$$\tilde{H}_0(\Delta) = \ker \partial_0 / \text{im } \partial_1 = \ker \partial_0 = \text{Span} \{ \bar{2} - \bar{1}, \bar{3} - \bar{1}, \bar{4} - \bar{1} \}$$

$$\tilde{\beta}_0 = 3.$$

Question D.15. *What are the homology groups and Betti numbers for $\Delta = \{ \bar{1}, \dots, \bar{n} \}$ for general $n \geq 1$?*

I.2.

$$\begin{array}{cccc}
 \bar{5} & \bar{6} & & \\
 \bullet & \bullet & & \\
 | & | & & \\
 \bullet & \bullet & \bullet & \bullet \\
 \bar{1} & \bar{2} & \bar{3} & \bar{4}
 \end{array}$$

$$0 \rightarrow \mathbb{R}^2 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{R}^6 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{R} \rightarrow 0$$

We have

$$\dim(\partial_0) = 1, \text{nullity}(\partial_0) = 6 - 1 = 5$$

$$\dim(\partial_1) = 2, \text{nullity}(\partial_1) = 0.$$

Therefore, $\tilde{\beta}_0 = 5 - 2 = 3$ and $\tilde{\beta}_1 = 0$. The same as in example I.

Homology:

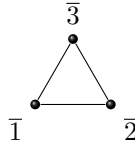
$$\tilde{H}_0 = \ker \partial_0 / \text{im } \partial_1 = \text{Span} \{ \bar{2} - \bar{1}, \bar{3} - \bar{1}, \bar{4} - \bar{1}, \bar{5} - \bar{1}, \bar{6} - \bar{1} \} / \text{Span} \{ \bar{5} - \bar{1}, \bar{6} - \bar{2} \}$$

$$\simeq \text{Span} \{ \bar{2} - \bar{1}, \bar{3} - \bar{1}, \bar{4} - \bar{1} \}$$

In $\tilde{H}_0$, we have $\bar{5} = \bar{1}$ and $\bar{6} = \bar{2}$, which means, $\bar{5} - \bar{1} = 0$ and $\bar{6} - \bar{1} = \bar{2} - \bar{1}$.

Question D.16. *How does this example generalize?*

II.1



$$0 \rightarrow \mathbb{R}^3 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}} \mathbb{R}^3 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} \mathbb{R} \rightarrow 0$$

We have

$$\dim(\partial_0) = 1, \text{nullity}(\partial_0) = 3 - 1 = 2$$

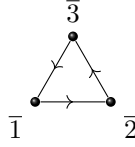
$$\dim(\partial_1) = 2, \text{nullity}(\partial_1) = 3 - 2 = 1.$$

Therefore, $\tilde{\beta}_0 = 2 - 2 = 0$ and $\tilde{\beta}_1 = 1$.

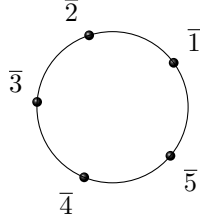
Homology:

$$\tilde{H}_1 = \ker \partial_1 / \text{im } \partial_2 = \ker \partial_1 = \text{Span}\{\overline{23} - \overline{13} + \overline{12}\}.$$

A picture of the generator for $\tilde{H}_1$:



II.2.



$$0 \rightarrow \mathbb{R}^5 \xrightarrow[\partial_1]{\begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}} \mathbb{R}^5 \xrightarrow[\partial_0]{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{R} \rightarrow 0$$

We have

$$\dim(\partial_0) = 1, \text{nullity}(\partial_0) = 5 - 1 = 4$$

$$\dim(\partial_1) = 4, \text{nullity}(\partial_1) = 5 - 4 = 1.$$

Therefore, $\tilde{\beta}_0 = 4 - 4 = 0$ and $\tilde{\beta}_1 = 1$.

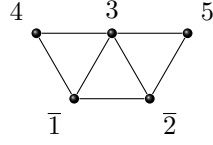
Homology:

$$\tilde{H}_1 = \ker \partial_1 / \text{im } \partial_2 = \ker \partial_1 = \text{Span}\{\overline{12} + \overline{23} + \overline{34} + \overline{45} - \overline{15}\}$$

The first homology is generated by a cycle of edges.

Question D.17. What happens in homology if we start with the triangle and subdivide its edges arbitrarily?

III.1.



Let's compute the first homology.

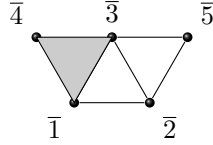
$$0 \xrightarrow{\partial_2} \mathbb{R}^7 \xrightarrow[\partial_1]{} \mathbb{R}^5$$

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We have $\dim \partial_1 = 4$, and so $\text{nullity} \partial_1 = 7 - 4 = 3$. Therefore $\tilde{\beta}_1 = 3$. The homology is generated by the cycles surrounding the three bounded faces of the complex as drawn above:

$$\tilde{H}_1 = \ker \partial_1 = \text{Span}\{\overline{23} - \overline{13} + \overline{12}, \overline{34} - \overline{14} + \overline{13}, \overline{35} - \overline{25} + \overline{23}\}$$

III.2.



For first homology, note that $\partial_2(\overline{134}) = \overline{34} - \overline{14} + \overline{13}$. We get

$$\mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^7 \xrightarrow[\partial_1]{} \mathbb{R}^5$$

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We have $\text{nullity}(\partial_1) = 7 - 4 = 3$ and $\dim \partial_2 = 1$. Therefore, $\tilde{\beta}_1 = 3 - 1 = 2$. The homology is generated by the cycles surrounding the two unfilled bounded faces of the complex:

$$\begin{aligned} \tilde{H}_1 = \ker \partial_1 \text{ im } \partial_2 &= \text{Span}\{\overline{23} - \overline{13} + \overline{12}, \overline{34} - \overline{14} + \overline{13}, \overline{35} - \overline{25} + \overline{23}\} / \text{Span}\{\overline{34} - \overline{14} + \overline{13}\} \\ &\simeq \text{Span}\{\overline{23} - \overline{13} + \overline{12}, \overline{35} - \overline{25} + \overline{23}\} \end{aligned}$$

The cycle $\overline{34} - \overline{14} + \overline{13}$ is the boundary of the shaded face, and thus has become 0 in the homology group.

Question D.18. *How does this example generalize?*

Exercise D.19. Draw the following simplicial complexes, determine their Betti numbers, and describe bases for their homology groups. Recall that a *facet* of a simplicial complex is a face that is maximal with respect to inclusion. So we can describe a simplicial complex by just listing its facets. The whole simplicial complex then consists of the facets and all of their subsets.

- (1) Δ with facets $\overline{123}$, $\overline{24}$, $\overline{34}$, $\overline{45}$, $\overline{56}$, $\overline{57}$, $\overline{89}$, $\overline{10}$.
- (2) Δ with facets $\overline{123}$, $\overline{14}$, $\overline{24}$, and $\overline{34}$.
- (3) Δ with facets $\overline{123}$, $\overline{124}$, $\overline{134}$, $\overline{234}$, $\overline{125}$, $\overline{135}$, $\overline{235}$. (Two hollow tetrahedra glued along a face.)

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