This is an untimed exam. You may use our course notes, lecture slides, your homework problems, and the homework solutions (on our Moodle page) but **no other outside resources**. Please submit your work via Gradescope by 1:10 p.m., Friday, October 13. If you find any typos, please let me know by email (davidp@reed.edu). If any typos are found, I will send email to the class.

PROBLEM 1. Let $M := \mathbb{P}^1 \times \mathbb{P}^1$. We can refer to points in M by a pair of homogeneous coordinates: (s,t) for the first factor and (u,v) for the second. Define the mapping

$$\begin{aligned} g\colon \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3 \\ ((s,t),(u,v)) &\mapsto (su, sv, tu, tv). \end{aligned}$$

(a) Describe an atlas for M consisting of four charts:

$$(U_{su}, h_{su}), (U_{sv}, h_{sv}), (U_{tu}, h_{tu}), (U_{tv}, h_{tv}),$$

where, for instance, $U_{su} = \{((s,t),(u,v)) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid s \neq 0, u \neq 0\}$. Define all of these sets and their chart mappings.

(b) Find the transition functions
  (i) from U_{su} to U_{tu} and
  (ii) from U_{sv} to U_{tu}.

(c) Show that g is well-defined: why is it independent of the choice of bi-homogeneous coordinates, and why is its image a point in $\mathbb{P}^3$ (note that $(0,0,0,0) \notin \mathbb{P}^3$)?

(d) Let the homogeneous coordinates on $\mathbb{P}^3$ be (a,b,c,d). Begin the job of showing that g is smooth by describing g in local coordinates with respect to U_{su} on M and (U_a, h_a) on $\mathbb{P}^3$ (where U_a is the set of points in $\mathbb{P}^3$ such that $a \neq 0$ and h_a is the usual coordinate mapping).

PROBLEM 2. The n-sphere is the set $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. It has the topology induced by $\mathbb{R}^{n+1}$, i.e., a set is open in S^n if and only if it is the intersection of an open set of $\mathbb{R}^{n+1}$ with S^n. Each point in S^n has some non-zero coordinate. For $i = 1, \dots, n+1$, define $U_i^+ = \{x \in S^n : x_i > 0\}$ and $U_i^- = \{x \in S^n : x_i < 0\}$. Define $\pi_i(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$. Then the collection of charts (U_i^+, π_i) and (U_i^-, π_i) for $i = 1, \dots, n+1$ serves as an atlas for S^n.

(a) To verify that these charts are differentiably related,
  (i) compute the transition function from U_1^+ to U_2^+, and
  (ii) compute the transition function from U_1^+ to U_2^-.

  The smoothness of the rest of the transition functions follows by symmetry.

(b) Consider the function $f(x) = \sum_{i=1}^{n+1} x_i^4$ defined on S^n. Let $p \in S^n$ with $p_1 > 0$. With respect to the chart (U_1^+, π_1), take the local coordinates to be $x_2, \dots, x_{n+1}$, as seems natural in this case. Let $a, b \in \mathbb{R}$, and let

$$v := a \left(\frac{\partial}{\partial x_2} \right)_p + b \left(\frac{\partial}{\partial x_3} \right)_p$$

be a tangent vector at p. Calculate $v(f)$. In other words, think of v as a derivation and apply it to f. (Don't worry if the answer is not beautiful.)

PROBLEM 3. Let M be an n-dimensional manifold. For each $p \in M$, let ξ_p be the $\mathbb{R}$-algebra of germs of functions at p. The germs vanishing at p are denoted by

$$\mathfrak{m}_p := \{f \in \xi_p \mid f(p) = 0\} \subset \xi_p.$$

(Recall that the value of $f \in \xi_p$ at p is well-defined; so in particular, the notion of a germ being zero at p is well-defined). Note that $\mathfrak{m}_p$ is an *ideal* in ξ_p, meaning that it is closed under addition and has the property that if $f \in \mathfrak{m}_p$ and $g \in \xi_p$, then $fg \in \mathfrak{m}_p$. Next, define

$$\mathfrak{m}_p^2 := \{\textstyle\sum_{i=1}^k f_i g_i \mid k \in \mathbb{N} \text{ and } f_i, g_i \in \mathfrak{m}_p \text{ for } i = 1, \dots k\}.$$

We think of $\mathfrak{m}_p^2$ as the germs "vanishing to order two" at p.

Now $\mathfrak{m}_p$ is a real vector space, and $\mathfrak{m}_p^2$ is a vector subspace of $\mathfrak{m}_p$. So we can consider the quotient vector space

$$\mathfrak{m}_p/\mathfrak{m}_p^2,$$

which is just the space of germs vanishing at p, except we consider two such germs to be the same if they differ by an element of $\mathfrak{m}_p^2$. In particular, elements of $\mathfrak{m}_p^2$ are treated as 0 in $\mathfrak{m}_p/\mathfrak{m}_p^2$. The purpose of this exercise is to show

$$\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^* \approx T_pM.$$

(Thus, it also follows that $\mathfrak{m}_p/\mathfrak{m}_p^2 \approx T_p^*M$, which provides an interesting way of thinking of cotangent space.)

(a) Think of T_pM as the space of derivations of germs and define

$$\begin{aligned} \alpha\colon (\mathfrak{m}_p/\mathfrak{m}_p^2)^* &\to T_pM \\ \phi &\mapsto \alpha(\phi) \end{aligned}$$

where

$$\begin{aligned} \alpha(\phi)\colon \xi_p &\to \mathbb{R} \\ f &\mapsto \phi(f - f(p)). \end{aligned}$$

(Note that $f - f(p)$ vanishes at p, so it may be considered as an element of $\mathfrak{m}_p/\mathfrak{m}_p^2$.) Linearity of both α and $\alpha(\phi)$ is straightforward and does not need to be checked here. Prove that $\alpha(\phi)$ is a derivation. Hint:

$$fg - f(p)g(p) = (f - f(p))(g - g(p)) + f(p)(g - g(p)) + g(p)(f - f(p)).$$

(b) Now define

$$\begin{aligned} \beta\colon T_pM &\to (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \\ v &\mapsto \beta(v) \end{aligned}$$

where

$$\begin{aligned} \beta(v)\colon \mathfrak{m}_p/\mathfrak{m}_p^2 &\to \mathbb{R} \\ f &\mapsto v(f). \end{aligned}$$

(i) Show that $\beta(v)$ is well-defined.
(ii) Show that α and β are inverses.

Problem 4. Let $\mathbb{R}^{(\omega)} := \bigoplus_{i=1}^{\infty} \mathbb{R}$ be the collection of all sequences of real numbers with only a finite number of nonzero terms. Let $\mathbb{R}^{\omega} := \prod_{i=1}^{\infty} \mathbb{R}$ be the collection of all sequences of real numbers. The standard basis vectors $\{e_i\}_{i=0}^{\infty}$ form basis for $\mathbb{R}^{(\omega)}$. A basis for $\mathbb{R}^{\omega}$ would be much harder to describe. (Note that the all-ones vector $(1, 1, \dots)$ is in $\mathbb{R}^{\omega}$ but is not a linear combination of the e_i. Linear combinations have, by definition, a finite number of summands.) We would like to show $\mathbb{R}^{(\omega)}$ and $\mathbb{R}^{\omega}$ are the categorical coproduct and product, respectively, in the category of vector spaces. For instance, first consider $\mathbb{R}^{(\omega)}$. For $i = 1, 2, \dots$, there are canonical injections $\ell_i \colon \mathbb{R} \to \mathbb{R}^{(\omega)}$ sending $x \in \mathbb{R}$ to the sequence whose i-th term is x and whose other terms are zeroes. Suppose X is a real vector space and you are given linear mappings $f_i \colon \mathbb{R} \to X$ for each i.

(a) Show there is a unique linear mapping g so that the following diagram commutes for each i:

$$\begin{tikzcd} X & \mathbb{R}^{(\omega)} \arrow[l, dotted, "\exists! g"'] \\ & \mathbb{R} \arrow[ul, "f_i"] \arrow[u, "\ell_i"'] \end{tikzcd}$$

The mapping g is usually denoted $\oplus_i f_i \colon \mathbb{R}^{(\omega)} \to X$.

(b) To show that $\mathbb{R}^{\omega}$ is the product, you need to show the "dual" result, turning all the arrows around. There are canonical projections $\pi_i \colon \mathbb{R}^{\omega} \to \mathbb{R}$ sending a sequence to its i-th term. Show that given linear mappings $f_i \colon X \to \mathbb{R}$ for each i, there exists a unique linear mapping g so that the following diagram commutes for each i:

$$\begin{tikzcd} X \arrow[r, dotted, "\exists! g"] \arrow[dr, "f_i"'] & \mathbb{R}^{\omega} \arrow[d, "\pi_i"] \\ & \mathbb{R} \end{tikzcd}$$

The mapping g in this case is usually denoted $\prod_i f_i \colon X \to \mathbb{R}^{\omega}$.

(c) Show that $(\mathbb{R}^{(\omega)})^* \approx \mathbb{R}^{\omega}$ by providing linear isomorphisms

$$\alpha \colon (\mathbb{R}^{(\omega)})^* \to \mathbb{R}^{\omega}$$

$$\beta \colon \mathbb{R}^{\omega} \to (\mathbb{R}^{(\omega)})^*$$

and showing that $\alpha \circ \beta = \mathrm{id}_{\mathbb{R}^{\omega}}$ and $\beta \circ \alpha = \mathrm{id}_{(\mathbb{R}^{(\omega)})^*}$.[1]

[1] Thus, we have an example of a vector space V for which V^* is not isomorphic to its dual. [It is impossible to have a linear isomorphism between $\mathbb{R}^{(\omega)}$ and $\mathbb{R}^{\omega}$ since only one has countable dimension.] Recall that $V^* \approx V$ whenever V is finite-dimensional.